Coset Space Dimensional Reduction and Wilson Flux Breaking of Ten-Dimensional $\mathcal{N} = 1$, E_8 Gauge Theory

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Abstract

We consider a $\mathcal{N} = 1$ supersymmetric E_8 gauge theory, defined in ten dimensions and we determine all four-dimensional gauge theories resulting from the generalized dimensional reduction a la Forgacs-Manton over coset spaces, followed by a subsequent application of the Wilson flux spontaneous symmetry breaking mechanism. Our investigation is constrained only by the requirements that (i) the dimensional reduction leads to the potentially phenomenologically interesting, anomaly free, four-dimensional E_6, SO_{10} and SU_5 GUTs and (ii) the Wilson flux mechanism makes use only of the freely acting discrete symmetries of all possible six-dimensional coset spaces.

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Contents

1 Introduction **2**

2 Coset Space Dimensional Reduction **5**

2.1 Coset space geometry 5

2.2 Reduction of a D-dimensional Yang-Mills-Dirac Lagrangian 7

2.3 The four-dimensional theory 11

2.4 Remarks on Grand Unified Theories resulting from CSDR 12

3 Wilson flux breaking mechanism in CSDR **12**

3.1 Wilson flux breaking mechanism 13

3.2 Further remarks concerning the use of the $F^{S/R}$ 14

3.3 Symmetry breaking patterns of E_6-like GUTs 14

3.3.1 The $\mathbb{Z}_2$ case 15

3.3.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case 18

3.4 Symmetry breaking pattern of $SO(10)$-like GUTs 19

3.4.1 The $\mathbb{Z}_2$ case 19

3.4.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case. 20

4 Classification of semi-realistic particle physics models **21**

4.1 Dimensional reduction over symmetric coset spaces 21

4.1.1 Reduction of $G = E_8$ using $\frac{SO(7)}{SO(6)}$ 22

4.1.2 Reduction of $G = E_8$ using $\frac{SU(4)}{SU(3)\times U(1)}$ 24

4.1.3 Reduction of $G = E_8$ using $\frac{SU(3)}{SU(2)\times U(1)} \times \frac{SU(2)}{U(1)}$ 24

4.1.4 Reduction of $G = E_8$ using $\left(\frac{SU(2)}{U(1)}\right)^3$ 26

4.1.5 Reduction of $G = E_8$ using $\frac{Sp(4)}{SU(2)\times SU(2)} \times \frac{SU(2)}{U(1)}$ 28

4.2 Dimensional reduction over non-symmetric coset spaces 29

4.2.1 Reduction of $G = E_8$ using $\frac{G_2}{SU(3)}$ 29

4.2.2 Reduction of $G = E_8$ using $\left(\frac{Sp(4)}{SU(2)\times U(1)}\right)_{nonmax}$ 30

4.2.3 Reduction of $G = E_8$ using $\frac{SU(3)}{U(1)\times U(1)}$ 31

5 Conclusions **33**

Appendices **34**

A Dimensional reduction over symmetric $6D$ coset spaces **34**

B Dimensional reduction over non-symmetric $6D$ coset spaces **39**

1 Introduction

The LEP era has established the Standard Model (SM) as the present day front of our knowledge concerning the theory of Elementary Particle Physics. On the other hand, already the plethora of its free parameters is suggesting the existence of physics beyond the SM that could explain at least some of them and possibly reduce them. The celebrated proposal of grand unification has inspired the particle physics community since it was providing an interesting reduction of couplings in the gauge sector of the SM. However, given that most of the free parameters of the SM is related to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory, it is natural to search for frameworks that could unify them. Then various schemes, with the Coset Space Dimensional Reduction (CSDR) [1–14] being pioneer, were suggesting that a unification of gauge and Higgs sectors can be achieved in higher dimensions. The four-dimensional gauge and Higgs fields are simply the surviving components of the gauge fields of a pure gauge theory defined in higher dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions. A major achievement in this direction is the possibility to obtain chiral theories in four dimensions [10, 11]. A final step towards unification of the gauge and fermions introduced in the higher dimensional theory is to demand that they are members of the same vector supermultiplet of a higher dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. Then another achievement is that the CSDR over non-symmetric cosets leads to softly broken supersymmetric theories [14] with all parameters determined (at the classical level). The latter is in a very interesting contrast to the usual supersymmetric extensions of the SM, where the soft supersymmetry breaking sector introduces a huge number of new free parameters.

Concerning supersymmetry, the nature of the four-dimensional theory depends on the corresponding nature of the compact space used to reduce the higher dimensional theory. Specifically the reduction over CY spaces leads to supersymmetric theories [15] in four dimensions, the reduction

over symmetric coset spaces leads to non-supersymmetric theories, while a reduction over non-symmetric ones leads to softly broken supersymmetric theories [14].

In the spirit described above a very welcome additional input is that string theory suggests furthermore the dimension and the gauge group of the higher dimensional supersymmetric theory [15]. Further support to this unified description comes from the fact that the reduction of the theory over coset [3] and CY spaces [15] provides the four-dimensional theory with scalars belonging in the fundamental representation (rep.) of the gauge group as are introduced in the SM. In addition the fact that the SM is a chiral theory lead us to consider D-dimensional supersymmetric gauge theories with $D = 4n + 2$ [3,11], which include the ten dimensions suggested by the heterotic string theory [15].

For many years the studies on the reduction of string theories had as a dominant direction those that consider the CY spaces as describing the higher compact dimensions. However one should note that there exist some problems too, mostly due to the complicated geometry of CY spaces. For instance their metric is not known explicitly, while their Euler characteristic is usually too large to predict an acceptable number of fermion generations. Moreover, in Calabi-Yau compactifications the resulting low-energy field theory in four dimensions contains a number of massless chiral fields, characteristic of the internal geometry, known as moduli. These fields correspond to flat directions of the effective potential and therefore their values are left undetermined. Since these values specify the masses and couplings of the four-dimensional theory, the theory has limited predictive power.

Fortunately, the moduli problem in the form described above appears only in the simplest choice of string backgrounds, where out of the plethora of closed-string fields only the metric is assumed to be non-trivial. By considering more general backgrounds involving "fluxes" [16, 17] as well as non-perturbative effects [18, 19], the four-dimensional theory can be provided with potentials for some or all moduli. The terminology "fluxes" refers to the inclusion of non-vanishing field strengths for the ten-dimensional antisymmetric tensor fields with directions purely inside the internal manifold.

The presence of fluxes has a dramatic impact on the geometry of the compactification space. Specifically, the energy carried by the fluxes back-reacts on the geometry of the internal space and the latter is deformed away from Ricci-flatness. Then, the CY manifolds used so often in string theory compactifications cease to be true solutions of the theory. For example, the requirement that some supersymmetry is preserved implies that the internal manifold is a non-Kähler space for heterotic strings with NS-NS fluxes [20–22], while it can be a non-complex manifold for type IIA strings [23–25].

A considerable amount of literature has been devoted to the problem of including appropriately the back-reaction of the fluxes on the internal manifold and constructing examples of manifolds which are true solutions of the theory. In general, these manifolds have non-vanishing torsion. Consequently, demanding that the low-energy theory is supersymmetric implies that the internal manifold admits a G-structure [26]. The existence of a G-structure is a generalization of the condition of special holonomy.

The case of SU(3)-structures is of special interest since the structure group SO(6) of the internal space can be reduced down to SU(3) in a way that a single spinor can be globally defined on it. This spinor is not necessarily covariantly constant with respect to the Levi-Civita connection, as in the case of Calabi-Yau manifolds, but it can be constant with respect to a torsionful connection. This

condition allows for a wider class of internal spaces, such as nearly-Kähler and half-flat manifolds. The Heterotic String theory has been recently studied in this context in [27]. Simple G-structure manifolds are six dimensional cosets possessing an $SU(3)$-structure. They were identified as supersymmetric solutions, e.g. in [28, 29] for the case of type II theories. In the Heterotic Supergravity cosets were introduced by [30] and recently studied in [31, 32]. Particularly, in [32] it was shown that supersymmetric compactifications of the Heterotic String theory of the form $AdS_4 \times S/R$ exist when background fluxes and general condensates are present. In addition, effective theories where constructed in [29, 33] in the case of type II supergravity. For a complete list of references see [34].

Due to the above developments and given that the non-symmetric six-dimensional coset spaces are nearly-Kähler we plan a detailed investigation of the CSDR of the heterotic string in two directions. The first concerns the supergravity sector [35], while the second deals with the gauge sector.

Here we limit ourselves in the study of the CSDR of the ten-dimensional E_8 gauge theory under certain conditions. Specifically in the present work, starting with an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory defined in ten dimensions we classify the semi-realistic particle physics models resulting from their CSDR and a subsequent application of Wilson flux spontaneous symmetry breaking. The space-time on which the theory is defined can be written in the compactified form $M^4 \times B$, with M^4 the ordinary Minkowski spacetime and $B = S/R$ a six-dimensional homogeneous coset space. We constrain our investigation in those cases that the dimensional reduction leads in four dimensions to phenomenologically interesting and anomaly-free Grand Unified Theories (GUTs) such as E_6, $SO(10)$, $SU(5)$. However since, as already mentioned, the four-dimensional surviving scalars transform in the fundamental rep. of the resulting gauge group are not suitable for the GUT breaking towards the SM. As a way out has been suggested [3, 36] to take advantage of non-trivial topological properties of the extra compactification coset space, apply the Hosotani-Wilson flux breaking mechanism [37, 38] and break the gauge symmetry of the theory further. The second constraint that we impose in our investigation is that the discrete symmetries, which we employ when we apply the Wilson flux mechanism, act freely on all possible six-dimensional coset spaces, i.e. after we mode out the discrete symmetries from a given coset space, there are no points in the resulting space that remain invariant. This is an obvious requirement to the extent that we deal with field theory which we assume in the present work. Our main objective is the investigation to which extent applying both methods namely CSDR and Wilson flux breaking mechanism, one can obtain reasonable low energy models.

In section (sec.) 2 we present the CSDR scheme in sufficient detail to make the paper self-contained. We recall some elements of the coset space geometry (sec. 2.1), the principle of the CSDR scheme and the constraints that the surviving fields of the four-dimensional theory have to obey (secs 2.2, 2.3), and we finally make some remarks on the GUTs that come from the CSDR scheme (sec. 2.4). In sec. 3 we recall the Wilson flux breaking mechanism and pave the way for the full investigation which is presented in sec 4. More specifically, after recalling the mechanism itself (sec. 3.1), we comment on the freely acting discrete symmetries of the coset spaces we use, which potentially lead to models with phenomenological interest (sec. 3.2). In secs 3.3 and 3.4 we determine the topologically induced symmetry breaking patterns of the GUTs of our present interest. In sec. 4 we present our investigation and a complete list of our results, on which we comment in sec. 5.

2 Coset Space Dimensional Reduction

Given a gauge theory defined in higher dimensions the obvious way to dimensionally reduce it is to demand that the field dependence on the extra coordinates is such that the Lagrangian is independent of them. A crude way to fulfill this requirement is to discard the field dependence on the extra coordinates, while an elegant one is to allow for a non-trivial dependence on them, but impose the condition that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Then the Lagrangian will be independent of the extra coordinates just because it is gauge invariant. This is the basis of the CSDR scheme [1–3], which assumes that B is a compact coset space, S/R.

In the CSDR scheme one starts with a Yang-Mills-Dirac Lagrangian, with gauge group G, defined on a D-dimensional spacetime M^D, with metric g^{MN}, which is compactified to $M^4 \times S/R$ with S/R a coset space. The metric is assumed to have the form

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix}, \tag{1}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$ and g^{ab} is the coset space metric. The requirement that transformations of the fields under the action of the symmetry group of S/R are compensated by gauge transformations lead to certain constraints on the fields. The solution of these constraints provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. Therefore a potential unification of all low energy interactions, gauge, Yukawa and Higgs is achieved, which was the first motivation of this framework.

It is interesting to note that the fields obtained using the CSDR approach are the first terms in the expansion of the D-dimensional fields in harmonics of the internal space S/R. The effective field theories resulting from compactification of higher dimensional theories contain also towers of massive higher harmonics (Kaluza-Klein) excitations, whose contributions at the quantum level alter the behavior of the running couplings from logarithmic to power [39]. As a result the traditional picture of unification of couplings may change drastically [40]. Higher dimensional theories have also been studied at the quantum level using the continuous Wilson renormalization group [41, 42] which can be formulated in any number of space-time dimensions with results in agreement with the treatment involving massive Kaluza-Klein excitations.

In the following we give a short description of the CSDR scheme, the constraints which have to be satisfied by the field content of the theory and recall how a four-dimensional gauge theory of unconstrained fields can be obtained. Complete reviews can be found in [3, 43].

2.1 Coset space geometry

To recall some aspects of the coset space geometry, we divide the generators of S, Q_A in two sets: the generators of R, Q_i $(i = 1, \ldots, \dim R)$, and the generators of S/R, $Q_a(a = \dim R+1 \ldots, \dim S)$, and $\dim(S/R) = \dim S - \dim R = d$. Then the commutation relations for the generators of S are

the following

$$[Q_i, Q_j] = f_{ij}{}^k Q_k\,, \tag{2a}$$

$$[Q_i, Q_a] = f_{ia}{}^b Q_b\,, \tag{2b}$$

$$[Q_a, Q_b] = f_{ab}{}^i Q_i + f_{ab}{}^c Q_c\,. \tag{2c}$$

So S/R is assumed to be a reductive but in general non-symmetric coset space. When S/R is symmetric, the $f_{ab}{}^c$ in (2c) vanish. Let us call the coordinates of $M^4 \times S/R$ space $x^M = (x^\mu, y^\alpha)$, where α is a curved index of the coset, a is a tangent space index and y defines an element of S which is a coset representative, $L(y)$. The vielbein and the R-connection are defined through the Maurer-Cartan form which takes values in the Lie algebra of S

$$L^{-1}(y)dL(y) = e^A_\alpha Q_A dy^\alpha\,. \tag{3}$$

Using (3) we can compute that at the origin $y = 0$, $e^a_\alpha = \delta^a_\alpha$ and $e^i_\alpha = 0$ which is a usefull result in order to determine the constraints that the four-dimensional matter fields have to obey, as we recall in sec. 2.2.

A connection on S/R which is described by a connection-form $\theta^a{}_b$, has in general torsion and curvature. In the general case where torsion may be non-zero, we calculate first the torsionless part $\omega^a{}_b$ by setting the torsion form T^a equal to zero,

$$T^a = de^a + \omega^a{}_b \wedge e^b = 0\,, \tag{4}$$

while using the Maurer-Cartan equation,

$$de^a = \frac{1}{2} f^a{}_{bc} e^b \wedge e^c + f^a{}_{bi} e^b \wedge e^i\,, \tag{5}$$

we see that the condition of having vanishing torsion is solved by

$$\omega^a{}_b = -f^a{}_{ib} e^i - D^a{}_{bc} e^c\,, \tag{6}$$

where

$$D^a{}_{bc} = \frac{1}{2} g^{ad} [f_{db}{}^e g_{ec} + f_{cb}{}^e g_{de} - f_{cd}{}^e g_{be}]\,.$$

Note that the connection-form $\omega^a{}_b$ is S-invariant. This means that parallel transport commutes with the S action [43].

In the case of non-vanishing torsion we have

$$T^a = de^a + \theta^a{}_b \wedge e^b\,, \tag{7}$$

where

$$\theta^a{}_b = \omega^a{}_b + \tau^a{}_b\,,$$

with

$$\tau^a{}_b = -\frac{1}{2} \Sigma^a{}_{bc} e^c\,, \tag{8}$$

while the contorsion $\Sigma^a{}_{bc}$ is given by

$$\Sigma^a{}_{bc} = T^a{}_{bc} + T_{bc}{}^a - T_{cb}{}^a \tag{9}$$

in terms of the torsion components $T^a{}_{bc}$. Therefore in general and for the case of non-symmetric cosets the connection-form $\theta^a{}_b$ is

$$\theta^a{}_b = -f^a{}_{ib}e^i - \left(D^a{}_{bc} + \frac{1}{2}\Sigma^a{}_{bc}\right)e^c = -f^a{}_{ib}e^i - G^a{}_{bc}e^c\,. \tag{10}$$

The natural choice of torsion which would generalize the case of equal radii [30,44–46], $T^a{}_{bc} = \eta f^a{}_{bc}$ would be $T^a{}_{bc} = 2\tau D^a{}_{bc}$ except that the D's do not have the required symmetry properties. Therefore we must define Σ as a combination of D's which makes Σ completely antisymmetric and S-invariant according to the definition given above. Thus we are led to the definition

$$\Sigma_{abc} \equiv 2\tau(D_{abc} + D_{bca} - D_{cba})\,. \tag{11}$$

By choosing vanishing parameter τ in the eqs (11) and (10) above we obtain the *Riemannian connection*, $\theta_R{}^a{}_b = -f^a{}_{ib}e^i - D^a{}_{bc}e^c$. On the other hand, by adjusting the radii and τ we can obtain the *canonical connection*, $\theta_C{}^a{}_b = -f^a{}_{bi}e^i$ which is an R-gauge field [30]. In general though the $\theta^a{}_b$ connection in its general form is an $SO(6)$ field, i.e. lives on the tangent space of the six-dimensional cosets we consider and describes their general holonomy. In sec. 2.3 we will show how the $G^{ab}{}_c$ term of eq. (10) it is connected with the geometrical and torsion contributions that the masses of the surviving four-dimensional gaugini acquire. Since we are interested here in four-dimensional models without light supersymmetric particles we keep $\theta^a{}_b$ general.

2.2 Reduction of a D-dimensional Yang-Mills-Dirac Lagrangian

The group S acts as a symmetry group on the extra coordinates. The CSDR scheme demands that an S-transformation of the extra d coordinates is a gauge transformation of the fields that are defined on $M^4 \times S/R$, thus a gauge invariant Lagrangian written on this space is independent of the extra coordinates.

To see this in detail we consider a D-dimensional Yang-Mills-Dirac theory with gauge group G defined on a manifold M^D which as stated will be compactified to $M^4 \times S/R$, $D = 4 + d$, $d = \dim S - \dim R$

$$A = \int d^4x d^d y \sqrt{-g}\Big[-\frac{1}{4}Tr\left(F_{MN}F_{K\Lambda}\right)g^{MK}g^{N\Lambda} + \frac{i}{2}\overline{\psi}\Gamma^M D_M\psi\Big]\,, \tag{12}$$

where

$$D_M = \partial_M - \theta_M - A_M\,, \tag{13}$$

with

$$\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda} \tag{14}$$

the spin connection of M^D, and

$$F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N]\,, \tag{15}$$

where M, N run over the D-dimensional space. The fields A_M and ψ are, as explained, symmetric in the sense that any transformation under symmetries of S/R is compensated by gauge transformations. The fermion fields can be in any rep. F of G unless a further symmetry is required.

Here since we assume dimensional reductions of $\mathcal{N}=1$ supersymmetric gauge theory the higher dimensional fermions have to transform in the adjoint of higher dimensional gauge group. To be more specific let ξ^α_A, $A=1,\ldots,\mathrm{dim}S$, be the Killing vectors which generate the symmetries of S/R and W_A the compensating gauge transformation associated with ξ_A. Define next the infinitesimal coordinate transformation as $\delta_A \equiv L_{\xi_A}$, the Lie derivative with respect to ξ, then we have for the scalar, vector and spinor fields,

$$\begin{aligned}
\delta_A\phi &= \xi^\alpha_A\partial_\alpha\phi = D(W_A)\phi\,,\\
\delta_A A_\alpha &= \xi^\beta_A\partial_\beta A_\alpha + \partial_\alpha\xi^\beta_A A_\beta = \partial_\alpha W_A - [W_A, A_\alpha]\,,\\
\delta_A\psi &= \xi^\alpha_A\psi - \frac{1}{2}G_{Abc}\Sigma^{bc}\psi = D(W_A)\psi\,.
\end{aligned} \tag{16}$$

W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the appropriate reps of the fields. G_{Abc} represents a tangent space rotation of the spinor fields. The variations δ_A satisfy, $[\delta_A,\delta_B]={f_{AB}}^C\delta_C$ and lead to the following consistency relation for W_A's,

$$\xi^\alpha_A\partial_\alpha W_B - \xi^\alpha_B\partial_\alpha W_A - [W_A,W_B] = {f_{AB}}^C W_C\,. \tag{17}$$

Furthermore the W's themselves transform under a gauge transformation [3] as,

$$W_A^{(g)} = g\,W_A\,g^{-1} + (\delta_A g)g^{-1}\,. \tag{18}$$

Using (18) and the fact that the Lagrangian is independent of y we can do all calculations at $y=0$ and choose a gauge where $W_a=0$.

The detailed analysis of the constraints (16) given in refs [1,3] provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. Here we give the results. The components $A_\mu(x,y)$ of the initial gauge field $A_M(x,y)$ become, after dimensional reduction, the four-dimensional gauge fields and furthermore they are independent of y. In addition one can find that they have to commute with the elements of the R_G subgroup of G. Thus the four-dimensional gauge group H is the centralizer of R in G, $H=C_G(R_G)$. Similarly, the $A_\alpha(x,y)$ components of $A_M(x,y)$ denoted by $\phi_\alpha(x,y)$ from now on, become scalars at four dimensions. These fields transform under R as a vector v, i.e.

$$\begin{gathered}
S \supset R\\
\mathrm{adj}\,S = \mathrm{adj}\,R + \mathrm{v}\,.
\end{gathered} \tag{19}$$

Moreover $\phi_\alpha(x,y)$ act as an intertwining operator connecting induced representations (reps) of R acting on G and S/R. This implies, exploiting Schur's lemma, that the transformation properties of the fields $\phi_\alpha(x,y)$ under H can be found if we express the adjoint rep. of G in terms of $R_G\times H$

$$\begin{gathered}
G \supset R_G\times H\\
\mathrm{adj}\,G = (\mathrm{adj}\,R,1) + (1,\mathrm{adj}\,H) + \sum(r_i,h_i)\,.
\end{gathered} \tag{20}$$

Then if $\mathrm{v}=\sum s_i$, where each s_i is an irreducible representation (irrep.) of R, there survives an h_i multiplet for every pair (r_i,s_i), where r_i and s_i are identical irreps of R.

Turning next to the fermion fields [3, 9–11, 47–50] similarly to scalars, they act as intertwining operators between induced reps acting on G and the tangent space of S/R, $SO(d)$. Proceeding

along similar lines as in the case of scalars to obtain the rep. of H under which the four-dimensional fermions transform, we have to decompose the rep. F of the initial gauge group in which the fermions are assigned under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i)\,, \tag{21}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j\,. \tag{22}$$

Then for each pair t_i and σ_i, where t_i and σ_i are irreps there is an h_i multiplet of spinor fields in the four-dimensional theory. In order however to obtain chiral fermions in the effective theory we have to impose further requirements. We first impose the Weyl condition in D dimensions. In $D = 4n + 2$ dimensions which is the case at hand, the decomposition of the left handed, say spinor under $SU(2) \times SU(2) \times SO(d)$ is

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \overline{\sigma}_d)\,. \tag{23}$$

Furthermore in order to be $\sigma_d \neq \overline{\sigma}_d$ the coset space S/R must be such that rank(R) = rank(S) [3,51]. The six-dimensional coset spaces which satisfy this condition are listed in the first column of tables 1 and 2. Then under the $SO(d) \supset R$ decomposition we have

$$\sigma_d = \sum \sigma_k\,, \qquad \overline{\sigma}_d = \sum \overline{\sigma}_k\,. \tag{24}$$

Case	**$6D$ Coset Spaces**	**$\mathbf{Z}(S)$**	**W**	**V**	**F**
a	$\frac{SO(7)}{SO(6)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6} \leftrightarrow \mathbf{6}$	$\mathbf{4} \leftrightarrow \overline{\mathbf{4}}$
b	$\frac{SU(4)}{SU(3)\times U(1)}$	$\mathbb{Z}_4$	$\mathbb{1}$	$\mathbf{6} = \mathbf{3}_{(-2)} + \overline{\mathbf{3}}_{(2)}$ $-$	$\mathbf{4} = \mathbf{1}_{(3)} + \mathbf{3}_{(-1)}$ $-$
c	$\frac{Sp(4)}{(SU(2)\times U(1))_{max}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6} = \mathbf{3}_{(-2)} + \mathbf{3}_{(2)}$ $\mathbf{3}_{(-2)} \leftrightarrow \mathbf{3}_{(2)}$	$\mathbf{4} = \mathbf{1}_{(3)} + \mathbf{3}_{(-1)}$ $\mathbf{1}_{(3)} \leftrightarrow \mathbf{1}_{(-3)}\quad \mathbf{3}_{(-1)} \leftrightarrow \mathbf{3}_{(1)}$
d	$\left(\frac{SU(3)}{SU(2)\times U(1)}\right) \times \left(\frac{SU(2)}{U(1)}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbf{6}{=}\mathbf{1}_{(0,2a)} + \mathbf{1}_{(0,-2a)}$ $+\mathbf{2}_{(b,0)} + \mathbf{2}_{(-b,0)}$ $\mathbf{1}_{(0,2a)} \leftrightarrow \mathbf{1}_{(0,-2a)}$	$\mathbf{4} = \mathbf{2}_{(0,a)} + \mathbf{1}_{(b,-a)} + \mathbf{1}_{(-b,-a)}$ $\mathbf{2}_{(0,a)} \leftrightarrow \mathbf{2}_{(0,-a)}$ $\mathbf{1}_{(b,-a)} \leftrightarrow \mathbf{1}_{(b,a)}$ $\mathbf{1}_{(-b,-a)} \leftrightarrow \mathbf{1}_{(-b,a)}$
e	$\left(\frac{Sp(4)}{SU(2)\times SU(2)}\right) \times \left(\frac{SU(2)}{U(1)}\right)$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2$	$\mathbf{6} = (\mathbf{2},\mathbf{2})_{(0)} + (\mathbf{1},\mathbf{1})_{(2)} + (\mathbf{1},\mathbf{1})_{(-2)}$ ($\mathbb{Z}_2$ of $SU(2)/U(1)$) $(\mathbf{1},\mathbf{1})_{(2)} \leftrightarrow (\mathbf{1},\mathbf{1})_{(-2)}$	$\mathbf{4} = (\mathbf{2},\mathbf{1})_{(1)} + (\mathbf{1},\mathbf{2})_{(-1)}$ $(\mathbf{2},\mathbf{1})_{(1)} \leftrightarrow (\mathbf{2},\mathbf{1})_{(-1)}$ $(\mathbf{1},\mathbf{2})_{(1)} \leftrightarrow (\mathbf{1},\mathbf{2})_{(-1)}$
f	$\left(\frac{SU(2)}{U(1)}\right)^3$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^3$	$\mathbf{6}{=}(2a,0,0) + (0,2b,0) + (0,0,2c)$ $+(-2a,0,0) + (0,-2b,0) + (0,0,-2c)$ each $\mathbb{Z}_2$ changes the sign of a,b, c	$\mathbf{4}{=}(a,b,c) + (-a,-b,c)$ $+(-a,b,-c) + (a,-b,-c)$ each $\mathbb{Z}_2$ changes the sign of a,b, c

Table 1: **Six-dimensional symmetric cosets spaces with** rank(R) = rank(S)**.** *The freely acting discrete symmetries* Z(S) *and* W *for each case are listed. The transformation properties of the vector and spinor representations under R are also noted.*

In the following sections we assume that the higher dimensional theory is $\mathcal{N} = 1$ supersymmetric. Therefore the higher dimensional fermion fields have to be considered transforming in the adjoint of E_8 which is vectorlike. In this case each term (t_i, h_i) in eq. (21) will be either self-conjugate or it will have a partner $(\overline{t}_i, \overline{h}_i)$. According to the rule described in eqs (21), (22) and considering σ_d we will have in four dimensions left-handed fermions transforming as $f_L = \sum h^L_k$. It is important

Case	$6D$ Coset Spaces	$\mathbf{Z}(S)$	W	V	F
a'	$\frac{G_2}{SU(3)}$	**1**	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{3}+\overline{\mathbf{3}}$ $\mathbf{3}\leftrightarrow\overline{\mathbf{3}}$	$\mathbf{4}=\mathbf{1}+\mathbf{3}$ $\mathbf{1}\leftrightarrow\mathbf{1}$ $\mathbf{3}\leftrightarrow\overline{\mathbf{3}}$
b'	$\frac{Sp(4)}{(SU(2)\times U(1))_{nonmax}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{1}_{(2)}+\mathbf{1}_{(-2)}+\mathbf{2}_{(1)}+\mathbf{2}_{(-1)}$ $\mathbf{1}_{(2)}\leftrightarrow\mathbf{1}_{(-2)}$ $\mathbf{2}_{(1)}\leftrightarrow\mathbf{2}_{(-1)}$	$\mathbf{4}=\mathbf{1}_{(0)}+\mathbf{1}_{(2)}+\mathbf{2}_{(-1)}$ $\mathbf{1}_{(2)}\leftrightarrow\mathbf{1}_{(-2)}$ $\mathbf{1}_{(0)}\leftrightarrow\mathbf{1}_{(0)}$ $\mathbf{2}_{(1)}\leftrightarrow\mathbf{2}_{(-1)}$
c'	$\frac{SU(3)}{U(1)\times U(1)}$	$\mathbb{Z}_3$	$\mathbf{S}_3$	$\mathbf{6}=(a,c)+(b,d)+(a+b,c+d)$ $+(-a,-c)+(-b,-d)$ $+(-a-b,-c-d)$	$\mathbf{4}=(0,0)$ $+(a,c)+(b,d)+(-a-b,-c-d)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(-b,-d)$ $(a+b,c+d)\leftrightarrow(a,c)$ $(-a,-c)\leftrightarrow(-a-b,-c-d)$	$(b,d)\leftrightarrow(-b,-d)$ $(a,c)\leftrightarrow(a+b,c+d)$ $(-a-b,-c-d)\leftrightarrow(-a,-c)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(a+b,c+d)$ $(a,c)\leftrightarrow(-a,-c)$ $(-b,-d)\leftrightarrow(-a-b,-c-d)$	$(b,d)\leftrightarrow(a+b,c+d)$ $(a,c)\leftrightarrow(-a,-c)$ $(-a-c,-b-d)\leftrightarrow(-b,-d)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(-a,-c)$ $(a+b,c+d)\leftrightarrow(-a-b,-c-d)$ $(a,c)\leftrightarrow(-b,-d)$	$(b,d)\leftrightarrow(-a,-c)$ $(a,c)\leftrightarrow(-b,-d)$ $(-a-b,-c-d)\leftrightarrow(a+b,c+d)$

Table 2: **Six-dimensional non-symmetric cosets spaces with** rank(R) = rank(S)**.** *The available freely acting discrete symmetries* Z(S) *and* W *for each case are listed. The transformation properties of vector and spinor representations under* R *are also noted.*

to notice that since σ_d is non self-conjugate, f_L is non self-conjugate too. Similarly from $\overline{\sigma}_d$ we will obtain the right handed rep. $f_R = \sum \overline{h}_k^R$ but as we have assumed that F is vector-like, $\overline{h}_k^R \sim h_k^L$. Therefore there will appear two sets of Weyl fermions with the same quantum numbers under H. This is already a chiral theory but still one can go further and try to impose the Majorana condition in order to eliminate the doubling of the fermion spectrum. However this is not required in the present case of study, where we apply the Hosotani mechanism for the further breaking of the gauge symmetry, as we will explain in sec. 3.

An important requirement is that the resulting four-dimensional theories should be anomaly free. Starting with an anomaly free theory in higher dimensions, Witten [52] has given the condition to be fulfilled in order to obtain anomaly free four-dimensional theories. The condition restricts the allowed embeddings of R into G by relating them with the embedding of R into $SO(6)$, the tangent space of the six-dimensional cosets we consider [3, 53]. To be more specific if $Ł_a$ are the generators of R into G and T_a are the generators of R into $SO(6)$ the condition reads

$$Tr(Ł_a Ł_b) = 30\, Tr(T_a T_b)\,. \tag{25}$$

According to ref. [53] the anomaly cancellation condition (25) is automatically satisfied for the choice of embedding

$$E_8 \supset SO(6) \supset R\,, \tag{26}$$

which we adopt here. Furthermore concerning the abelian group factors of the four-dimensional gauge theory, we note that the corresponding gauge bosons surviving in four dimensions become massive at the compactification scale [6, 52] and therefore, they do not contribute in the anomalies; they correspond only to global symmetries.

2.3 The four-dimensional theory

Next let us obtain the four-dimensional effective action. Assuming that the metric is block diagonal, taking into account all the constraints and integrating out the extra coordinates we obtain in four dimensions the following Lagrangian

$$A = C \int d^4x \biggl(-\frac{1}{4} F^t_{\mu\nu} F^{t\mu\nu} + \frac{1}{2}(D_\mu \phi_a)^t (D^\mu \phi^a)^t + V(\phi) + \frac{i}{2}\overline{\psi}\Gamma^\mu D_\mu \psi - \frac{i}{2}\overline{\psi}\Gamma^a D_a \psi \biggr), \tag{27}$$

where $D_\mu = \partial_\mu - A_\mu$ and $D_a = \partial_a - \theta_a - \phi_a$ with $\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc}$ the connection of the coset space and Σ^{bc} the $SO(6)$ generators. With C we denote the volume of the coset space. The potential $V(\phi)$ is given by

$$V(\phi) = F_{ab}F^{ab} = -\frac{1}{4} g^{ac} g^{bd} Tr(f_{ab}{}^C \phi_C - [\phi_a, \phi_b])(f_{cd}{}^D \phi_D - [\phi_c, \phi_d]), \tag{28}$$

where, $A = 1, \ldots, \dim S$ and f ' s are the structure constants appearing in the commutators of the generators of the Lie algebra of S. The expression (28) for $V(\phi)$ is only formal because the ϕ_a must satisfy the constraints coming from eqs (16),

$$f_{ai}{}^c \phi_c - [\phi_a, \phi_i] = 0, \tag{29}$$

where the ϕ_i generate R_G. These constraints imply that some components ϕ_a's are zero, some are constants and the rest can be identified with the genuine Higgs fields according with the rules presented in eqs (19) and (20).

When $V(\phi)$ is expressed in terms of the unconstrained independent Higgs fields, it remains a quartic polynomial which is invariant under gauge transformations of the final gauge group H, and its minimum determines the vacuum expectation values of the Higgs fields [5, 7, 8]. The minimization of the potential is in general a difficult problem. However, when S has an isomorphic image S_G in G, the four-dimensional gauge group H will break spontaneously to a subgroup K, which is the centralizer of S_G in the group of the higher dimensional theory G, $K = C_G(S_G)$ [3, 4]. This can be illustrated as follows:

$$\begin{array}{rcl} G \supset S_G & \times & K \\ \cup & & \cap \\ R & \times & H\,. \end{array} \tag{30}$$

Furthermore when ϕ acquires v.e.v. the $V(\phi)$ vanishes. It should be stressed that, in this class of models, the four-dimensional fermions acquire large masses due to geometrical contributions at the compactification scale [3, 50]. In general it can be proven [3] that dimensional reduction over a symmetric coset space always gives a potential of spontaneous breaking form which is not the case of non-symmetric cosets of more than one radii.

In the fermion part of the Lagrangian the first term is just the kinetic term of fermions, while the second is the Yukawa term [50, 54]. The last term in (27) can be written as

$$L_D = -\frac{i}{2}\overline{\psi}\Gamma^a \left(\partial_a - \frac{1}{2} f_{ibc} e^i_\gamma e^\gamma_a \Sigma^{bc} - \frac{1}{2} G_{abc}\Sigma^{bc} - \phi_a \right)\psi = \frac{i}{2}\overline{\psi}\Gamma^a \nabla_a \psi + \overline{\psi} V \psi, \tag{31}$$

where

$$\nabla_a = -\partial_a + \frac{1}{2} f_{ibc} e^i_\gamma e^\gamma_a \Sigma^{bc} + \phi_a \,, \tag{32}$$

$$V = \frac{i}{4} \Gamma^a G_{abc} \Sigma^{bc} \,, \tag{33}$$

where G_{abc} is given in eq. (10) as $G^a{}_{bc} = D^a{}_{bc} + \frac{1}{2}\Sigma^a{}_{bc}$. The CSDR constraints tell us that $\partial_a \psi = 0$. Furthermore we can consider the Lagrangian at the point $y = 0$, due to its invariance under S-transformations, and according to the discussion in sec. 2.1 $e^i_\gamma = 0$ at that point. Therefore (32) becomes just $\nabla_a = \phi_a$ and the term $\frac{i}{2}\overline{\psi}\Gamma^a \nabla_a \psi$ in eq. (31) is exactly the Yukawa term. The last term of eq. (31) vanishes in the case of dimensional reduction over symmetric cosets, whereas in the case of non-symmetric cosets is responsible for the masses of the four-dimensional gaugini [14]. However, as explained in sec. 2.1, this mass term can be suitably modified under appropriate adjustment of the torsion and the radii of the non-symmetric coset in question.

2.4 Remarks on Grand Unified Theories resulting from CSDR

Here we make few remarks on models resulting from the coset space dimensional reduction of an $\mathcal{N} = 1$, E_8 gauge theory which is defined on a ten-dimensional compactified space $M^D = M^4 \times (S/R)$. The coset spaces S/R we consider are listed in the first column of tables 1 and 2. In order to obtain four-dimensional GUTs potentially with phenomenological interest, namely $\mathcal{H} = E_6$, $SO(10)$ and $SU(5)$, is sufficient to consider only embeddings of the isotropy group R of the coset space in

$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SU(3) \,, \qquad \text{for} \quad \mathcal{H} = E_6 \,, \tag{34a}$$

$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SO(6) \sim SU(4) \,, \qquad \text{for} \quad \mathcal{H} = SO(10) \,, \tag{34b}$$

$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SU(5) \,, \qquad \text{for} \quad \mathcal{H} = SU(5) \,. \tag{34c}$$

As it was noted in sec. 2.2 the anomaly cancellation condition (25) is satisfied automatically for the choice of embedding

$$E_8 \supset SO(6) \supset R \,, \tag{35}$$

which we adopt here. This requirement is trivially fullfiled for the case of $R \hookrightarrow \mathcal{R}$ embeddings of eq. (34b) which lead to $SO(10)$ GUTs in four dimensions. It is obviously also satisfied for the case of $R \hookrightarrow \mathcal{R}$ embeddings of eq. (34a) since $SU(3) \subset SO(6)$. The above case leads to E_6 GUTs in four dimensions. Finally, $R \hookrightarrow \mathcal{R}$ embeddings of eq. (34c) are excluded since the requirement (35) cannot be satisfied.

3 Wilson flux breaking mechanism in CSDR

The surviving scalars in a four-dimensional GUT, being in the fundamental rep. of the gauge group are not able to provide the appropriate superstrong symmetry breaking towards the standard model. As a way out it has been suggested [36] to take advantage of non-trivial topological properties of the

compactification coset space, apply the Hosotani or Wilson flux breaking mechanism [37, 38] and break the gauge symmetry of the theory further. Application of this mechanism imposes further constraints in the scheme.

In the next subsections we first recall the Wilson flux breaking mechanism, we make some remarks on specific cases which potentially lead to interesting models and we finally calculate the actual symmetry breaking patterns of the GUTs.

3.1 Wilson flux breaking mechanism

Let us briefly recall the Wilson flux mechanism for breaking spontaneously a gauge theory. Then instead of considering a gauge theory on $M^4 \times B_0$, with B_0 a simply connected manifold, and in our case a coset space $B_0 = S/R$, we consider a gauge theory on $M^4 \times B$, with $B = B_0/F^{S/R}$ and $F^{S/R}$ a freely acting discrete symmetry¶ of B_0. It turns out that B becomes multiply connected, which means that there will be contours not contractible to a point due to holes in the manifold. For each element $g \in F^{S/R}$, we pick up an element U_g in H, i.e. in the four-dimensional gauge group of the reduced theory, which can be represented as the Wilson loop

$$U_g = \mathcal{P}exp\left(-i \int_{\gamma_g} T^a A^a_M(x) dx^M\right) , \tag{36}$$

where $A^a_M(x)$ are vacuum H fields with group generators T^a, γ_g is a contour representing the abstract element g of $F^{S/R}$, and $\mathcal{P}$ denotes the path ordering.

Now if γ_g is chosen not to be contractible to a point, then $U_g \neq 1$ although the vacuum field strength vanishes everywhere. In this way an homomorphism of $F^{S/R}$ into H is induced with image T^H, which is the subgroup of H generated by $\{U_g\}$. A field $f(x)$ on B_0 is obviously equivalent to another field on B_0 which obeys $f(g(x)) = f(x)$ for every $g \in F^{S/R}$. However in the presence of the gauge group H this statement can be generalized to

$$f(g(x)) = U_g f(x) \,. \tag{37}$$

Next, one would like to see which gauge symmetry is preserved by the vacuum. The vacuum has $A^a_\mu = 0$ and we represent a gauge transformation by a space-dependent matrix $V(x)$ of H. In order to keep $A^a_\mu = 0$ and leave the vacuum invariant, $V(x)$ must be constant. On the other hand, $f \to Vf$ is consistent with equation (37), only if $[V, U_g] = 0$ for all $g \in F^{S/R}$. Therefore the H breaks towards the centralizer of T^H in H, $K' = C_H(T^H)$. In addition the matter fields have to be invariant under the diagonal sum

$$F^{S/R} \oplus T^H \,, \tag{38}$$

in order to satisfy eq. (37) and therefore survive in the four-dimensional theory.

¶By freely acting we mean that for every element $g \in F$, except the identity, there exists no points of B_0 that remain invariant.

3.2 Further remarks concerning the use of the $F^{S/R}$

The discrete symmetries $F^{S/R}$, which act freely on coset spaces $B_0 = S/R$ are the center of S, $\mathrm{Z}(S)$ and the $\mathrm{W} = \mathrm{W}_S/\mathrm{W}_R$, with W_S and W_R being the Weyl groups of S and R, respectively [3, 55]. The freely acting discrete symmetries, $F^{S/R}$, of the specific six-dimensional coset spaces under discussion are listed in the second and third column of tables 1 and 2. The $F^{S/R}$ transformation properties of the vector and spinor irreps under R are noted in the last two columns of the same tables.

Our approach is to embed the $F^{S/R}$ discrete symmetries into four-dimensional $H = E_6$ and $SO(10)$ gauge groups. We make this choice only for bookeeping reasons since, according to sec. 3.1, the actual topological symmetry breaking takes place in higher dimensions. Few remarks are in order. In both classes of models, namely E_6 and $SO(10)$ GUTs, the use of the discrete symmetry of the center of S, $\mathrm{Z}(S)$, cannot lead to phenomenologically interesting cases since various components of the irreps of the four-dimensional GUTs containing the SM fermions do not survive. The reason is that the irreps of H remain invariant under the action of the discrete symmetry, $\mathrm{Z}(S)$, and as a result the phase factors gained by the action of T^H cannot be compensated. Therefore the complete SM fermion spectrum cannot be invariant under $F^{S/R} \oplus T^H$ and survive. On the other hand, the use of the Weyl discrete symmetry can lead to better results. Models with potentially interesting fermion spectrum can be obtained employing at least one $\mathbb{Z}_2 \subset \mathrm{W}$. Then, the fermion content of the four-dimensional theory is found to transform in linear combinations of the two copies of the CSDR-surviving left-handed fermions. Details will be given in sec. 4. As we will discuss there employing $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \times \mathrm{Z}(S)$ can also lead to interesting models.

Therefore the interesting cases for further study are

$$F^{S/R} = \begin{cases} \mathbb{Z}_2 \subseteq \mathrm{W} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \times \mathrm{Z}(S)\,. \end{cases} \tag{39}$$

3.3 Symmetry breaking patterns of E_6-like GUTs

Here we determine the image, T^H, that each of the discrete symmetries of eq. (39) induces in the gauge group $H = E_6$. We consider embeddings of the $F^{S/R}$ discrete symmetries into abelian subgroups of E_6 and examine their topologically induced symmetry breaking patterns [38]. These are realized by a diagonal matrix U_g of unit determinant, which as explained in sec. 3.1, has to be homomorphic to the considered discrete symmetry. In fig. 1 we present those E_6 decompositions which potentially lead to the SM gauge group structure [56].

$$\begin{tikzcd}
 & F_4 \arrow[r] & SU(3)\times SU^c(3)\ (\mathbf{1}) & \\
E_6 \arrow[ur] \arrow[r] \arrow[d] & SO(10) \arrow[r] & SU(5) \arrow[r] & SU^w(2)\times SU^c(3)\ (\mathbf{2}) \\
SU^w(3)\times SU(3)\times SU^c(3)\ (\mathbf{3}) & & &
\end{tikzcd}$$

Figure 1: *E_6 decompositions leading potentially to SM gauge group structure.*

3.3.1 The $\mathbb{Z}_2$ case

Embedding (1): $\mathbb{Z}_2 \hookrightarrow SU(3)$ of $E_6 \supset F_4 \supset SU(3)\times SU^c(3)$. We consider the maximal subgroups of E_6 and the corresponding decomposition of fundamental and adjoint irreps

$$\begin{aligned}
&E_6 \supset F_4 \supset SU(3)\times SU^c(3)\\
&\mathbf{27}{=}(\mathbf{1},\mathbf{1})+(\mathbf{8},\mathbf{1})+(\mathbf{3},\mathbf{3})+(\overline{\mathbf{3}},\overline{\mathbf{3}})\,,\\
&\mathbf{78}{=}(\mathbf{8},\mathbf{1})+(\mathbf{3},\mathbf{3})+(\overline{\mathbf{3}},\overline{\mathbf{3}})+(\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{8})+(\mathbf{6},\overline{\mathbf{3}})+(\overline{\mathbf{6}},\mathbf{3})
\end{aligned} \tag{40}$$

and embed the $F^{S/R}=\mathbb{Z}_2$ discrete symmetry in the $SU(3)$ group factor above. There exist two distinct possibilities of embedding, either $\mathbb{Z}_2 \hookrightarrow U^I(1)$ which appears under the $SU(3) \supset SU(2)\times U^{II}(1) \supset U^I(1)\times U^{II}(1)$ decomposition or $\mathbb{Z}_2 \hookrightarrow U^{II}(1)$. Since the former is trivial, namely cannot break the $SU(3)$ appearing in eq. (40), only the latter is interesting for further investigation. This is realized as

$$U_g^{(1)} = diag(-1,-1,1)\,. \tag{41}$$

Indeed $(U_g^{(1)})^2 = \mathbb{1}_3$ as required by the $F^{S/R}\mapsto H$ homomorphism and $det(U_g^{(1)})=1$ since U_g is an H group element.

Then, the various components of the decomposition of $SU(3)$ irreps under $SU(2)\times U(1)$ acquire the underbraced phase factors in the following list

$$\begin{aligned}
&SU(3)\supset SU(2)\times U(1)\\
&\mathbf{3}{=}\underbrace{\mathbf{1}_{(-2)}}_{(+1)}+\underbrace{\mathbf{2}_{(1)}}_{(-1)}\,,\\
&\mathbf{6}{=}\underbrace{\mathbf{1}_{(-4)}}_{(+1)}+\underbrace{\mathbf{2}_{(-1)}}_{(-1)}+\underbrace{\mathbf{3}_{(2)}}_{(+1)}\,,\\
&\mathbf{8}{=}\underbrace{\mathbf{1}_{(0)}}_{(+1)}+\underbrace{\mathbf{3}_{(0)}}_{(+1)}+\underbrace{\mathbf{2}_{(-3)}}_{(-1)}+\underbrace{\mathbf{2}_{(3)}}_{(-1)}\,.
\end{aligned} \tag{42}$$

Consequently the various components of the decomposition of E_6 irreps (40) under $F_4 \supset SU(3)\times$

$SU^c(3) \supset (SU(2) \times U(1)) \times SU^c(3)$ acquire the underbraced phase factors in the following list

$$
\begin{aligned}
&E_6 \supset SU(2) \times SU^c(3) \times U(1) \\
&\mathbf{27} = \underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)} \\
&\quad + \underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)} + \underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(2)}}_{(+1)} + \underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)} , \\
&\mathbf{78} = \underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{8})_{(0)}}_{(+1)} \\
&\quad + \underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)} + \underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(2)}}_{(+1)} + \underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(-4)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{3})_{(4)}}_{(+1)} \\
&\quad + \underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)} \\
&\quad + \underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)} + \underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)} + \underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)} \\
&\quad + \underbrace{(\overline{\mathbf{3}},\mathbf{3})_{(-2)}}_{(+1)} + \underbrace{(\mathbf{3},\overline{\mathbf{3}})_{(+2)}}_{(+1)} .
\end{aligned}
\tag{43}
$$

According to the discussion in sec. 3.1 the four-dimensional gauge group after the topological breaking is given by $K' = C_H(T^H)$. Counting the number of singlets under the action of $U_g^{(1)}$ in the **78** irrep. above suggests that $K' = SO(10) \times U(1)$, a fact which subsequently is determined according to the following decomposition of the **78** irrep.

$$
\begin{aligned}
&E_6 \supset SO(10) \times U(1) \\
&\mathbf{27} = \underbrace{\mathbf{1}_{(-4)}}_{(+1)} + \underbrace{\mathbf{10}_{(-2)}}_{(+1)} + \underbrace{\mathbf{16}_{(1)}}_{(-1)} , \\
&\mathbf{78} = \underbrace{\mathbf{1}_{(0)}}_{(+1)} + \underbrace{\mathbf{45}_{(0)}}_{(+1)} + \underbrace{\mathbf{16}_{(-3)}}_{(-1)} + \underbrace{\overline{\mathbf{16}}_{(3)}}_{(-1)} .
\end{aligned}
\tag{44}
$$

It is interesting to note that although one would naively expect the E_6 gauge group to break further towards the SM one this is not the case. The singlets under the action of $U_g^{(1)}$ which occur in the adjoint irrep. of E_6 in eq. (43) add up to provide a larger final unbroken gauge symmetry, namely $SO(10) \times U(1)$.

Embedding (2): $\mathbb{Z}_2 \hookrightarrow SU(5)$ of $E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)$. Similarly, we consider the maximal subgroups of E_6 and the corresponding decomposition of the fundamental and adjoint irreps

$$
\begin{aligned}
&E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1) \\
&\mathbf{27} = \mathbf{1}_{(0,-4)} + \mathbf{5}_{(2,-2)} + \overline{\mathbf{5}}_{(-2,-2)} + \mathbf{1}_{(-5,1)} + \overline{\mathbf{5}}_{(3,1)} + \mathbf{10}_{(-1,1)} , \\
&\mathbf{78} = \mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{24}_{(0,0)} + \mathbf{1}_{(-5,-3)} + \mathbf{1}_{(5,3)} \\
&\quad + \mathbf{5}_{(-3,3)} + \overline{\mathbf{5}}_{(3,-3)} + \mathbf{10}_{(4,0)} + \overline{\mathbf{10}}_{(-4,0)} + \mathbf{10}_{(-1,-3)} + \overline{\mathbf{10}}_{(1,3)} .
\end{aligned}
\tag{45}
$$

Our choice is to embed the $\mathbb{Z}_2$ discrete symmetry in an abelian $SU(5)$ subgroup in a way that is realized by the diagonal matrix

$$U_g^{(2)} = diag(-1,-1,1,1,1)\,. \tag{46}$$

Then the various components of the $SU(5)$ irreps decomposed under the $SU(2)\times SU(3)\times U(1)$ decomposition acquire the underbraced phase factors in the following list

$$\begin{aligned}
&SU(5)\supset SU(2)\times SU(3)\times U(1)\\
&\mathbf{5}{=}\underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)}+\underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)}\,,\\
&\mathbf{10}{=}\underbrace{(\mathbf{1},\mathbf{1})_{(6)}}_{(+1)}+\underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(-4)}}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)}\,,\\
&\mathbf{24}{=}\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{8})_{(0)}}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{3})_{(-5)}}_{(-1)}+\underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(5)}}_{(-1)}\,.
\end{aligned} \tag{47}$$

It can be proven, along the lines of the previous case (**1**), that $U_g^{(2)}$ leads to the breaking $E_6 \to SU(2)\times SU(6)$

$$\begin{aligned}
&E_6\supset SU(2)\times SU(6)\\
&\mathbf{27}{=}\underbrace{(\mathbf{2},\overline{\mathbf{6}})}_{(-1)}+\underbrace{(\mathbf{1},\mathbf{15})}_{(+1)}\,,\\
&\mathbf{78}{=}\underbrace{(\mathbf{3},\mathbf{1})}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{35})}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{20})}_{(-1)}\,,
\end{aligned} \tag{48}$$

i.e. we find again an enhancement of the final gauge group as compared to the naively expected one.

Note that other choices of $\mathbb{Z}_2$ into $SU(5)$ embeddings either lead to trivial or to phenomenologically uninteresting results.

Embedding (3): $\mathbb{Z}_2 \hookrightarrow SU(3)$ of $E_6 \supset SU^w(3)\times SU(3)\times SU^c(3)$. We consider the maximal subgroup of E_6 and the corresponding decomposition of fundamental and adjoint irreps

$$\begin{aligned}
&E_6\supset SU^w(3)\times SU(3)\times SU^c(3)\\
&\mathbf{27}{=}(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})+(\mathbf{3},\mathbf{1},\mathbf{3})+(\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}})\,,\\
&\mathbf{78}{=}(\mathbf{8},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{8})+(\mathbf{3},\mathbf{3},\overline{\mathbf{3}})+(\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{3})\,.
\end{aligned} \tag{49}$$

We furthermore assume an $\mathbb{Z}_2 \hookrightarrow SU(3)$ embedding, which is realized by

$$U_g^{(3)} = (\mathbb{1}_3)\otimes diag(-1,-1,1)\otimes(\mathbb{1}_3)\,. \tag{50}$$

Although this choice of embedding is not enough to lead to the SM gauge group structure, our results will be usefull for the discussion of the $\mathbb{Z}_2{\times}\mathbb{Z}_2'$ case which is presented in sec. 3.3.2. With the choice of embedding realized by the eq. (50) the second $SU(3)$ decomposes under $SU(2)\times U(1)$ as in eq. (42) and leads to the breaking (48), as before. As was mentioned in case (**1**) the choice of embedding $\mathbb{Z}_2 \hookrightarrow U^I(1)$, which appears under the decomposition $SU(3)\supset SU(2)\times U^{II}(1)\supset U^I(1)\times U^{II}(1)$

Embedd.	U_g	$\mathbf{K}'$
1	$U_g^{(1)}$	$SO(10) \times U(1)$
2	$U_g^{(2)}$	$SU(2) \times SU(6)$
3	$\mathbb{1}_3 \otimes U_g^{(1)} \otimes \mathbb{1}_3$	$SU(2) \times SU(6)$

Table 3: **Embeddings of $\mathbb{Z}_2$ discrete symmetry in E_6 GUT and its symmetry breaking patterns.** $U_g^{(1)} = diag(-1,-1,1)$ *and* $U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

of eq. (49), cannot break the $SU(3)$ group factor and it is not an interesting case for further investigation.

In table 3 we summarize the above results, concerning the topologically induced symmetry breaking patterns of the E_6 gauge group.

3.3.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case

Embedding ($\mathbf{2'}$): $\mathbb{Z}_2 \hookrightarrow SO(10)$ and $\mathbb{Z}_2' \hookrightarrow SU(5)$ of $E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)$. Here we embed the $\mathbb{Z}_2$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetry in the $SU(5)$ appearing under the decomposition $E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)$ as in case (**2**) above. Furthermore we embed the $\mathbb{Z}_2'$ discrete symmetry in the $SO(10)$ as

$$U_g' = -\mathbb{1}_{10}\,. \tag{51}$$

This leads to the breaking $E_6 \supset SU(2) \times SU(6)$ as before but with the signs of the phase factors, which appear in eq. (48), being reversed under the action of $U_g^{(2)} U_g'$.

Embedding ($\mathbf{3'}$): $\mathbb{Z}_2 \hookrightarrow SU(3)$ and $\mathbb{Z}_2' \hookrightarrow SU^w(3)$ of $E_6 \supset SU^w(3) \times SU(3) \times SU^c(3)$. Here we embed the $\mathbb{Z}_2$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetry in the $SU(3)$ group factor appearing under the $E_6 \supset SU(3)^w \times SU(3) \times SU(3)^c$ as in case (**3**) above. Furthermore we embed the $\mathbb{Z}_2'$ discrete symmetry in the $SU(3)^w$ group factor in a similar way. Then the embedding ($\mathbf{3'}$), which we discuss here, is realized by considering an element of the E_6 gauge group

$$U_g' U_g^{(3)} = diag(-1,-1,1) \otimes diag(-1,-1,1) \otimes (\mathbb{1}_3)\,, \tag{52}$$

Embedd.	U_g	U'_g	$\mathbf{K}'$
$\mathbf{2'}$	$U_g^{(2)}$	$-\mathbb{1}_{10}$	$SU(2)\times SU(6)$
$\mathbf{3'}$	$U_g^{(1)}\otimes\mathbb{1}_3\otimes\mathbb{1}_3$	$\mathbb{1}_3\otimes U_g^{(1)}\otimes\mathbb{1}_3$	$SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)$

Table 4: **Embeddings of $\mathbb{Z}_2\times\mathbb{Z}'_2$ discrete symmetries in E_6 GUT and its symmetry breaking patterns.** $U_g^{(1)}=diag(-1,-1,1)$ *and* $U_g^{(2)}=diag(-1,-1,1,1,1)$ *as in text.*

which leads to the breaking $E_6\to SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)$ as it is clear from the following decomposition of **78** irrep.

$$
\begin{aligned}
&E_6\supset SO(10)\times U(1)\supset SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)\\
&E_6\supset SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)\\
&\mathbf{27}=\underbrace{(\mathbf{1},\mathbf{1},\mathbf{1})_{(4)}}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{1},\mathbf{6})_{(-2)}}_{(+1)}\\
&\quad+\underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})_{(1)}}_{(-1)}+\underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{(1)}}_{(-1)}\,,\\
&\mathbf{78}=\underbrace{(\mathbf{1},\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{3},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{3},\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{1},\mathbf{15})_{(0)}}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{2},\mathbf{6})_{(0)}}_{(-1)}\\
&\quad+\underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})_{(-3)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})_{(3)}}_{(-1)}+\underbrace{(\mathbf{1},\mathbf{2},\mathbf{4})_{(3)}}_{(-1)}+\underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{(-3)}}_{(-1)}\,.
\end{aligned}
\tag{53}
$$

In table 4 we summarize the above results, concerning the topologically induced symmetry breaking patterns of the E_6 gauge group.

3.4 Symmetry breaking pattern of $SO(10)$-like GUTs

Here we determine the image, T^H, that each of the discrete symmetries of eq. (39) induces in the gauge group $H=SO(10)$. We consider embeddings of the $F^{S/R}$ discrete symmetries into abelian subgroups of $SO(10)$ GUTs and examine their topologically induced symmetry breaking patterns. The interesting $F^{S/R}\hookrightarrow SO(10)$ embeddings are those which potentially lead to SM gauge group structure, i.e.

$$SO(10)\supset SU(5)\times U^{II}(1)\supset SU^w(2)\times SU^c(3)\times U^I(1)\times U^{II}(1)\,.$$

3.4.1 The $\mathbb{Z}_2$ case

Embedding (1): $\mathbb{Z}_2\hookrightarrow SU(5)$ of $SO(10)\supset SU(5)\times U(1)$. In the present case we assume the maximal subgroup of $SO(10)$

$$
\begin{aligned}
&SO(10)\supset SU(5)\times U^{II}(1)\\
&\mathbf{10}=\mathbf{5}_{(2)}+\overline{\mathbf{5}}_{(-2)}\,,\\
&\mathbf{16}=\mathbf{1}_{(-5)}+\overline{\mathbf{5}}_{(3)}+\mathbf{10}_{(-1)}\,,\\
&\mathbf{45}=\mathbf{1}_{(0)}+\mathbf{24}_{(0)}+\mathbf{10}_{(4)}+\overline{\mathbf{10}}_{(-4)}\,,
\end{aligned}
\tag{54}
$$

Embedd.	U_g	$\mathbf{K'}$
1	$U_g^{(2)}$	$SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)$

Table 5: **Embedding of $\mathbb{Z}_2$ discrete symmetry in $SO(10)$ GUT and its symmetry breaking pattern.**$U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

Embedd.	U_g	U'_g	$\mathbf{K'}$
$\mathbf{1'}$	$U_g^{(2)}$	$-\mathbb{1}_{10}$	$SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)$

Table 6: **Embedding of $\mathbb{Z}_2 \times \mathbb{Z}'_2$ discrete symmetries in $SO(10)$ GUT and its symmetry breaking pattern.** $U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

and embed a $\mathbb{Z}_2 \hookrightarrow SU(5)$ which is realized as in eq. (46). Then, the **5**, **10** and **24** irreps of $SU(5)$ under the $SU(5) \supset SU(2) \times SU(3) \times U(1)$ decomposition read as in eq. (47) and leads to the breaking $SO(10) \rightarrow SU^a(2) \times SU^b(2) \times SU(4)$ which is a Pati-Salam type model,

$$\begin{aligned}
&SO(10) \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\
&\mathbf{10} = \underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{6})}_{(+1)}, \\
&\mathbf{16} = \underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})}_{(+1)}, \\
&\mathbf{45} = \underbrace{(\mathbf{3},\mathbf{1},\mathbf{1})}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{3},\mathbf{1})}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{15})}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{2},\mathbf{6})}_{(-1)} .
\end{aligned} \tag{55}$$

Again we notice that although one would naively expect the $SO(10)$ gauge group to break towards SM, this is not the case.

For completeness in table 5 we present the above case.

3.4.2 The $\mathbb{Z}_2 \times \mathbb{Z}'_2$ case.

Embedding ($\mathbf{1'}$): $\mathbb{Z}_2 \hookrightarrow SU(5)$ and $\mathbb{Z}'_2 \hookrightarrow SO(10)$ of $SO(10) \supset SU(5) \times U(1)$. Note that a second $\mathbb{Z}_2$ cannot break the $K' = SU^a(2) \times SU^b(2) \times SU(4)$ further. However by choosing the non-trivial embedding $U'_g = -\mathbb{1}_{10}$ of $\mathbb{Z}_2$ in the $SO(10)$ the phase factors appearing in eq. (55) have their signs reversed under the action of $U_g^{(2)} U'_g$.

Again in table 6 we present the above case.

4 Classification of semi-realistic particle physics models

Here starting from an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory defined in ten dimensions, we provide a complete classification of the semi-realistic particle physics models resulting from CSDR of the original theory and a subsequent application of the Wilson flux breaking mechanism. According to our requirements in sec. 2.4 the dimensional reduction of this theory over the six-dimensional coset spaces, leads to anomaly free E_6 and $SO(10)$ GUTs in four dimensions. Recall also that the four-dimensional surviving scalars transform in the fundamental of the resulting gauge group and are not suitable for the superstrong symmetry breaking of these GUTs towards the SM. One way out was discussed in sec. 3, namely the Wilson flux breaking mechanism. In the present section we investigate to which extent applying both methods, CSDR and Wilson flux breaking mechanism one can obtain reasonable low energy models.

4.1 Dimensional reduction over symmetric coset spaces

We consider all the possible embeddings $E_8 \supset SO(6) \supset R$ for the six-dimensional *symmetric* coset spaces, S/R, listed in the first column of table 1[‖]. These embeddings are presented in fig. 2. It is worth noting that in all cases the dimensional reduction of the initial gauge theory leads to an $SO(10)$ GUT according to the concluding remarks in sec. 2.2. The result of our examination in the present section is that the additional use of the Wilson flux breaking mechanism leads to four-dimensional theories of Pati-Salam type. In the following sections 4.1.1 - 4.1.5 we present in some detail our examination and the corresponding results, which we summarize in tables 7 and 8 presented in appendix A[**].

$$\begin{tikzcd}
 & SU(4)\ (\mathbf{1}) & & \\
SO(6) \sim SU(4) \arrow[ur] \arrow[r] \arrow[dr] & SU(3)\times U(1)\ (\mathbf{2}) \arrow[r] & SU^a(2)\times U(1)\times U(1)\ (\mathbf{3}) \arrow[r] & U(1)\times U(1)\times U(1)\ (\mathbf{4}) \\
 & SU^a(2)\times SU^b(2)\times U(1)\ (\mathbf{5}) \arrow[r] & SU^a(2)\times U(1)\times U(1)\ (\mathbf{6}) \arrow[r] & U(1)\times U(1)\times U(1)\ (\mathbf{7})
\end{tikzcd}$$

Figure 2: *Possible $E_8 \supset SO(6) \supset R$ embeddings for the symmetric coset spaces, S/R, of table 1.*

[‖] We have excluded the study of dimensional reduction over the $Sp(4)/(SU(2) \times U(1))_{max}$ coset space which does not admit fermions.

[**] For convenience we label the cases examined in the following subsections as 'Case No.x' with the 'No' denoting the embedding$R \hookrightarrow E_8$ and the 'x' the coset space we use. The same label is also used in tables 7 and 8.

4.1.1 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = SO(7)/SO(6)$. (Case 1a)

We consider Weyl fermions belonging in the adjoint of $G = E_8$ and the embedding of $R = SO(6)$ into E_8 suggested by the decomposition

$$\begin{aligned} &E_8 \supset SO(16) \supset SO(6) \times SO(10) \\ &\mathbf{248} {=} (\mathbf{1}, \mathbf{45}) + (\mathbf{6}, \mathbf{10}) + (\mathbf{15}, \mathbf{1}) + (\mathbf{4}, \mathbf{16}) + (\overline{\mathbf{4}}, \overline{\mathbf{16}})\,. \end{aligned} \tag{56}$$

If only the CSDR mechanism was applied the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SO(6)) = SO(10)\,.$$

According to table 1, the $R = SO(6)$ content of vector and spinor of $B_0 = S/R = SO(7)/SO(6)$ is **6** and **4**, respectively. Then applying the CSDR rules (19), (20) and (21), (22) the four-dimensional theory would contain scalars transforming as **10** under the $H = SO(10)$ gauge group and two copies of chiral fermion belonging in the $\mathbf{16}_L$ of H.

Next we apply in addition the Wilson flux breaking mechanism discussed already in sec. 3 and take into account the various observations made there. The freely acting discrete symmetries, $F^{S/R}$, of the coset space $SO(7)/SO(6)$ (case '**a**' in table 1) are the Weyl, W $= \mathbb{Z}_2$ and the center of S, Z(S) $= \mathbb{Z}_2$. As it was explained in sec. 3.2 the use of Z(S) alone is excluded. On the other hand, according to the discussion in sec. 3.4.1 the W discrete symmetry leads to a four-dimensional theory with gauge group

$$K' = C_H(T^H) = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)\,.$$

Then according to eq. (38), the surviving field content has to be invariant under the combined action of the considered discrete symmetry itself, $F^{S/R}$, and its induced image in the H gauge group, T^H. Using the W $= \mathbb{Z}_2$ discrete symmetry, the decomposition of the irrep. **10** of $SO(10)$ under the K' gauge group is given in eq. (55),

$$\begin{array}{rcl} SO(10) & \supset & SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\ \mathbf{10} & = & \underbrace{(\mathbf{2}, \mathbf{2}, \mathbf{1})}_{(-1)} + \underbrace{(\mathbf{1}, \mathbf{1}, \mathbf{6})}_{(+1)}\,. \end{array} \tag{57}$$

Then, recalling that the vector $B_0 = SO(7)/SO(6)$ is invariant under the action of W (see table 1), we conclude that the four-dimensional theory contains scalars transforming according to

$$(\mathbf{1}, \mathbf{1}, \mathbf{6})$$

of K'. Similarly, the irrep. **16** of $SO(10)$ decomposes under the K' as

$$\begin{array}{rcl} SO(10) & \supset & SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\ \mathbf{16} & = & \underbrace{(\mathbf{2}, \mathbf{1}, \mathbf{4})}_{(-1)} + \underbrace{(\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})}_{(+1)}\,. \end{array} \tag{58}$$

In this case the spinor of the tangent space of $SO(7)/SO(6)$ decomposed under $R = SO(6)$ is obviously **4**. Then, since the W transformation property is $\mathbf{4} \leftrightarrow \overline{\mathbf{4}}$ (see table 1), the fermion content of the four-dimensional theory transforms as

$$(\mathbf{2}, \mathbf{1}, \mathbf{4})_L - (\mathbf{2}, \mathbf{1}, \mathbf{4})'_L \qquad \text{and} \qquad (\mathbf{1}, \mathbf{2}, \mathbf{4})_L + (\mathbf{1}, \mathbf{2}, \mathbf{4})'_L \tag{59}$$

under K'.

In the present case as far as the spontaneous symmetry breaking of the four-dimensional theory is concerned, both theorems mentioned in sec 2.3 are applicable. According to the first theorem mentioned there, dimensional reduction over the $SO(7)/SO(6)$ symmetric coset space leads to a four-dimensional potential with spontaneously symmetry breaking form. However, since the four-dimensional scalar fields transform as $(\mathbf{1}, \mathbf{1}, \mathbf{6})$ under the K' gauge group obtaining a v.e.v. break the $SU(3)$ colour. Therefore, employing the W discrete symmetry is not an interesting case for further investigation.

Next if we use the $\mathrm{W} \times \mathrm{Z}(S) = \mathbb{Z}_2 \times \mathbb{Z}_2$ discrete symmetry, the Wilson flux breaking mechanism leads again to the Pati-Salam gauge group, K' (see sec. 3.4.2). However in this case, all the underbraced phase factors of eqs (57) and (58) are multiplied by -1. Therefore the four-dimensional theory now contains scalars transforming according to

$$(\mathbf{2}, \mathbf{2}, \mathbf{1})$$

of K', and two copies of chiral fermions transforming as in eq. (59) but with the signs of the linear combinations reversed.

Concerning the spontaneous symmetry breaking of the latter model, we note that the isometry group of the coset, $SO(7)$, is embeddable in E_8 as

$$\begin{array}{rcl} E_8 \supset SO(7) \times & SO(9) & \\ \cup & \cap & \\ SO(6) \times & SO(10)\,, & \end{array}$$

and according to the second theorem mentioned in sec. 2.3, if only the CSDR mechanism was applied, the final gauge group would be

$$\mathcal{H} = C_{E_8}(SO(7)) = SO(9)\,.$$

In other words the $\mathbf{10}$ of $SO(10)$ would obtain v.e.v. leading to the spontaneous symmetry breaking

$$\begin{array}{rcl} SO(10) & \rightarrow & SO(9) \\ \mathbf{10} & = & \langle \mathbf{1} \rangle + \mathbf{9}\,. \end{array} \tag{60}$$

However now we employ the Wilson flux breaking mechanism which breaks the gauge symmetry further in higher dimensions. It is instructive to understand the spontaneous breaking indicated in eq. (60) in this context too. A straightforward examination of the gauge group structure and the reps of the scalars that are involved, suggests that the breaking indicated in eq. (57) is realized in the present context as

$$\begin{array}{rcl} SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) & \rightarrow & SU^{diag}(2) \times SU(4) \\ (\mathbf{2}, \mathbf{2}, \mathbf{1}) & = & \langle(\mathbf{1}, \mathbf{1})\rangle + (\mathbf{3}, \mathbf{1})\,, \end{array} \tag{61}$$

i.e. the final gauge group of the four-dimensional theory is

$$K = SU^{diag}(2) \times SU(4)\,.$$

Accordingly, the fermions transform as

$$(\mathbf{2}, \mathbf{4})_L + (\mathbf{2}, \mathbf{4})'_L \qquad \text{and} \qquad (\mathbf{2}, \mathbf{4})_L - (\mathbf{2}, \mathbf{4})'_L$$

under K.

4.1.2 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = SU(4)/(SU(3) \times U(1))$. (Case 2b)

We consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the embedding of $R = SU(3) \times U(1)$ into E_8 suggested by the decomposition[††]

$$\begin{aligned} &E_8 \supset SO(16) \supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset SU(3) \times U^I(1) \times SO(10) \\ &\quad E_8 \supset (SU(3) \times U^I(1)) \times SO(10) \\ &\quad \mathbf{248} = (\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{45})_{(0)} + (\mathbf{8},\mathbf{1})_{(0)} + (\mathbf{3},\mathbf{10})_{(-2)} + (\overline{\mathbf{3}},\mathbf{10})_{(2)} \\ &\qquad + (\mathbf{3},\mathbf{1})_{(4)} + (\overline{\mathbf{3}},\mathbf{1})_{(-4)} + (\mathbf{1},\mathbf{16})_{(-3)} + (\mathbf{1},\overline{\mathbf{16}})_{(3)} \\ &\qquad + (\mathbf{3},\mathbf{16})_{(1)} + (\overline{\mathbf{3}},\overline{\mathbf{16}})_{(-1)} \, . \end{aligned} \tag{62}$$

If only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU(3) \times U^I(1)) = SO(10) \Big(\times U^I(1) \Big) \, ,$$

where the additional $U(1)$ factor in the parenthesis corresponds to a global symmetry, according to the concluding remarks in sec. 2.2. The $R = SU(3) \times U^I(1)$ content of the vector and spinor of $B_0 = S/R = SU(4)/(SU(3) \times U^I(1))$ can be read in the last two columns of table 1. Then according to the CSDR rules, the theory would contain scalars belonging in the $\mathbf{10}_{(-2)}$, $\mathbf{10}_{(2)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(3)}$ and $\mathbf{16}_{L(-1)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space $SU(4)/(SU(3) \times U(1))$ are not included in the list (39) of those ones that are worth to be examined further.

4.1.3 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = SU(3)/(SU(2) \times U(1)) \times (SU(2)/U(1))$. (Cases 3d, 6d)

We consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the following decomposition

$$\begin{aligned} E_8 &\supset SO(16) \supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset (SU'(3) \times U^{II}(1)) \times SO(10) \\ &\supset (SU^a(2) \times U^I(1) \times U^{II}(1)) \times SO(10) \end{aligned} \tag{63}$$

or

$$\begin{aligned} E_8 &\supset SO(16) \supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset (SU^a(2) \times SU^b(2) \times U^{II}(1)) \times SO(10) \\ &\supset (SU^a(2) \times U^I(1) \times U^{II}(1)) \times SO(10) \end{aligned} \tag{64}$$

In both cases we can properly redefine the $U(1)$ charges, and consequently choose an embedding of $R = SU(2) \times U^I(1) \times U^{II}(1)$ into E_8 as follows

$$\begin{aligned} &E_8 \supset (SU^a(2) \times U^{I'}(1) \times U^{II'}(1)) \times SO(10) \\ &\mathbf{248} = (\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{3},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{45})_{(0,0)} \\ &\quad + (\mathbf{1},\mathbf{1})_{(-2b,0)} + (\mathbf{1},\mathbf{1})_{(2b,0)} + (\mathbf{2},\mathbf{1})_{(-b,2a)} + (\mathbf{2},\mathbf{1})_{(b,-2a)} \\ &\quad + (\mathbf{2},\mathbf{1})_{(-b,-2a)} + (\mathbf{2},\mathbf{1})_{(b,2a)} + (\mathbf{1},\mathbf{10})_{(0,-2a)} + (\mathbf{1},\mathbf{10})_{(0,2a)} \\ &\quad + (\mathbf{2},\mathbf{10})_{(b,0)} + (\mathbf{2},\mathbf{10})_{(-b,0)} + (\mathbf{1},\mathbf{16})_{(b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(-b,a)} \\ &\quad + (\mathbf{1},\mathbf{16})_{(-b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(b,a)} + (\mathbf{2},\mathbf{16})_{(0,a)} + (\mathbf{2},\overline{\mathbf{16}})_{(0,-a)} \, . \end{aligned} \tag{65}$$

[††]This decomposition is in accordance with the Slansky tables [56] but with opposite $U(1)$ charge.

Here, a and b are the $U(1)$ charges of vector and fermion content of the coset space $B_0 = S/R = SU(3)/(SU^a(2) \times U^{I'}(1)) \times (SU(2)/U^{II'}(1))$, shown in the last two columns of table 1 (case '**d**'). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU^a(2) \times U^{I'}(1) \times U^{II'}(1)) = SO(10) \Big(\times U^{I'}(1) \times U^{II'}(1) \Big) ,$$

where the additional $U(1)$ factors in the parenthesis correspond to global symmetries. According to the CSDR rules, the four-dimensional model contains scalars belonging in $\mathbf{10}_{(0,-2a)}$, $\mathbf{10}_{(0,2a)}$, $\mathbf{10}_{(b,0)}$ and $\mathbf{10}_{(-b,0)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(b,-a)}$, $\mathbf{16}_{L(-b,-a)}$ and $\mathbf{16}_{L(0,a)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space under discussion are the center of S, $\mathrm{Z(S)} = \mathbb{Z}_3 \times \mathbb{Z}_2$ and the Weyl symmetry, $\mathrm{W} = \mathbb{Z}_2$. Then according to the list (39) the interesting cases to be examined further are the following two.

In the first case we employ the $\mathrm{W} = \mathbb{Z}_2$ discrete symmetry which leads to a four-dimensional theory with gauge symmetry group

$$K' = C_H(T^H) = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \Big(\times U^{I'}(1) \times U^{II'}(1) \Big) .$$

Similarly to the case discussed in sec. 4.1.1, the surviving scalars transform as

$$(\mathbf{1},\mathbf{1},\mathbf{6})_{(b,0)} \qquad \text{and} \qquad (\mathbf{1},\mathbf{1},\mathbf{6})_{(-b,0)} \tag{66}$$

under K' which are the only ones that are invariant under the action of W (table 1). Furthermore, taking into account the W transformation properties listed in the last column of table 1, as well as the decomposition of $\mathbf{16}$ irrep of $SO(10)$ under $SU^{(i)}(2) \times SU^{(i)}(2) \times SU(4)$ [see eq. (58)], we conclude that the four-dimensional fermions transform as

$$\begin{gathered}
(\mathbf{2},\mathbf{1},\mathbf{4})_{L(b,-a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(b,-a)} , \\
(\mathbf{1},\mathbf{2},\mathbf{4})_{L(b,-a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(b,-a)} , \\
\begin{aligned}
&(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-b,-a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-b,-a)} , \qquad && (\mathbf{2},\mathbf{1},\mathbf{4})_{L(0,a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(0,a)} , \\
&(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-b,-a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-b,-a)} , \qquad && (\mathbf{1},\mathbf{2},\mathbf{4})_{L(0,a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(0,a)} ,
\end{aligned}
\end{gathered} \tag{67}$$

under K'.

Once more we have spontaneous symmetry breaking (since the coset space is symmetric) which breaks the $SU(3)$-colour (since the scalars transform as in (66) under the K' gauge group). Therefore, employing the W discrete symmetry is not an interesting case for further investigation.

In the second case we use the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the $\mathrm{W} \times \mathrm{Z}(S)$ combination of discrete symmetries. The surviving scalars of the four-dimensional theory belong in the $(\mathbf{2},\mathbf{2},\mathbf{1})_{(b,0)}$ and $(\mathbf{2},\mathbf{2},\mathbf{1})_{(-b,0)}$ of the K' gauge group which remains the same as before. The fermions, on the other hand, transform as those in eq. (67) but with the signs of the linear combinations reversed. The final gauge group after the spontaneous symmetry breaking of the theory is found to be

$$K = SU^{diag}(2) \times SU(4) \Big(\times U^{I'}(1) \times U^{II'}(1) \Big),$$

and its fermions transform as

$$
\begin{array}{cc}
& (\mathbf{2},\mathbf{4})_{(b,-a)} + (\mathbf{2},\mathbf{4})'_{(b,-a)} \,, \\
& (\mathbf{2},\mathbf{4})_{(b,-a)} - (\mathbf{2},\mathbf{4})'_{(b,-a)} \,, \\
(\mathbf{2},\mathbf{4})_{(-b,-a)} + (\mathbf{2},\mathbf{4})'_{(-b,-a)} \,, & (\mathbf{2},\mathbf{4})_{(0,a)} + (\mathbf{2},\mathbf{4})'_{(0,a)} \,, \\
(\mathbf{2},\mathbf{4})_{(-b,-a)} - (\mathbf{2},\mathbf{4})'_{(-b,-a)} \,, & (\mathbf{2},\mathbf{4})_{(0,a)} - (\mathbf{2},\mathbf{4})'_{(0,a)}
\end{array} \tag{68}
$$

under K.

4.1.4 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = (SU(2)/U(1))^3$. (Cases 4f, 7f)

We consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the following decomposition

$$
\begin{aligned}
E_8 \supset SO(16) &\supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset SU'(3) \times U^{III}(1) \times SO(10) \\
&\supset (SU^a(2) \times U^{II}(1) \times U^{III}(1)) \times SO(10) \\
&\supset SO(10) \times U^{I}(1) \times U^{II}(1) \times U^{III}(1)
\end{aligned} \tag{69}
$$

or

$$
\begin{aligned}
E_8 \supset SO(16) &\supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \\
&\supset (SU^a(2) \times SU^b(2) \times U^{III}(1)) \times SO(10) \\
&\supset SO(10) \times U^{I}(1) \times U^{II}(1) \times U^{III}(1)
\end{aligned} \tag{70}
$$

In both cases we can properly redefine the $U(1)$ charges, and consequently choose an embedding of $R = SU(2) \times U^I(1) \times U^{II}(1)$ into E_8 as follows

$$
\begin{aligned}
E_8 \ &\supset SO(10) \times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \\
\mathbf{248} &= \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)} + \mathbf{45}_{(0,0,0)} \\
&\quad + \mathbf{1}_{(-2a,2b,0)} + \mathbf{1}_{(2a,-2b,0)} + \mathbf{1}_{(-2a,-2b,0)} + \mathbf{1}_{(2a,2b,0)} \\
&\quad + \mathbf{1}_{(-2a,0,-2c)} + \mathbf{1}_{(2a,0,2c)} + \mathbf{1}_{(0,-2b,-2c)} + \mathbf{1}_{(0,2b,2c)} \\
&\quad + \mathbf{1}_{(-2a,0,2c)} + \mathbf{1}_{(2a,0,-2c)} + \mathbf{1}_{(0,-2b,2c)} + \mathbf{1}_{(0,2b,-2c)} \\
&\quad + \mathbf{10}_{(0,0,2c)} + \mathbf{10}_{(0,0,-2c)} + \mathbf{10}_{(0,2b,0)} + \mathbf{10}_{(0,-2b,0)} \\
&\quad + \mathbf{10}_{(2a,0,0)} + \mathbf{10}_{(-2a,0,0)} + \mathbf{16}_{(a,b,c)} + \overline{\mathbf{16}}_{(-a,-b,-c)} \\
&\quad + \mathbf{16}_{(-a,-b,c)} + \overline{\mathbf{16}}_{(a,b,-c)} + \mathbf{16}_{(-a,b,-c)} + \overline{\mathbf{16}}_{(a,-b,c)} \\
&\quad + \mathbf{16}_{(a,-b,-c)} + \overline{\mathbf{16}}_{(-a,b,c)} \,.
\end{aligned} \tag{71}
$$

Then, if only the CSDR mechanism was applied, the four-dimensional gauge group would be

$$
H = C_{E_8}(U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1)) = SO(10) \Big(\times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \Big) \,.
$$

The same comment as in the previous cases holds for the additional $U(1)$ factors in the parenthesis. The $R = U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1)$ content of vector and spinor of $B_0 = S/R = (SU(2)/U^{I'}(1)) \times (SU(2)/U^{II'}(1)) \times (SU(2)/U^{III'}(1))$ can be read in the last two columns of table 1. According to the CSDR rules then, the resulting four-dimensional theory would contain scalars belonging in $\mathbf{10}_{(2a,0,0)}$, $\mathbf{10}_{(-2a,0,0)}$, $\mathbf{10}_{(0,2b,0)}$, $\mathbf{10}_{(0,-2b,0)}$, $\mathbf{10}_{(0,0,2c)}$ and $\mathbf{10}_{(0,0,-2c)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(a,b,c)}$, $\mathbf{16}_{L(-a,-b,c)}$, $\mathbf{16}_{L(-a,b,-c)}$ and $\mathbf{16}_{L(a,-b,-c)}$ under the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $(SU(2)/U(1))^3 \sim (S^2)^3$ are the center of S, Z(S) $= (\mathbb{Z}_2)^3$ and the Weyl discrete symmetry, W $= (\mathbb{Z}_2)^3$. Then according to the list (39) the interesting cases to be examined further are the following.

First, let us mod out the $(S^2)^3$ coset space by the $\mathbb{Z}_2 \subset$ W and consider the multiple connected manifold $S^2/\mathbb{Z}_2 \times S^2 \times S^2$. Then, the resulting four-dimensional gauge group will be

$$K' = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \Big(\times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \Big) .$$

The four-dimensional theory will contain scalar which belong in

$$\begin{array}{ll} (\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,2b,0)} \, , & (\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,-2b,0)} \, , \\ (\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,0,2c)} \, , & (\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,0,-2c)} \end{array}$$

of K'; these are the only ones that are invariant under the action of the considered $\mathbb{Z}_2 \subset$ W. However, linear combinations between the two copies of the CSDR-surviving left-handed fermions have no definite properties under the abelian factors of the K' gauge group and they do not survive. As a result, the model is not an interesting case for further investigation.

Second, if we employ the $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset$ W discrete symmetry and consider the manifold $S^2/\mathbb{Z}_2 \times S^2/\mathbb{Z}_2 \times S^2$, the resulting four-dimensional theory has the same gauge group as before, i.e. K'. Similarly as before, scalars transforms as

$$(\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,0,2c)} \, , \qquad (\mathbf{1}, \mathbf{1}, \mathbf{6})_{(0,0,-2c)}$$

under K'. However, no fermions survive in the four-dimensional theory and the model is again not an interesting case to examine further.

Finally, if we employ the $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset$ W $\times$ Z(S) discrete symmetry, the four-dimensional theory contains scalars which belong in

$$\begin{array}{ll} (\mathbf{2}, \mathbf{2}, \mathbf{1})_{(0,2b,0)} \, , & (\mathbf{2}, \mathbf{2}, \mathbf{1})_{(0,-2b,0)} \, , \\ (\mathbf{2}, \mathbf{2}, \mathbf{1})_{(0,0,2c)} \, , & (\mathbf{2}, \mathbf{2}, \mathbf{1})_{(0,0,-2c)} \end{array}$$

of K' but no fermions. The model is again not an interesting case for further study.

Therefore although the above studied cases have been obtained using discrete symmetries which are included in the list (39), no fermion fields survive in the four-dimensional theory. The reason is that we employ here only a subgroup of the Weyl discrete symmetry W $= (\mathbb{Z}_2)^3$ and we cannot form linear combinations among the two copies of the CSDR-surviving left-handed fermions which are invariant under eq. (38). The use of the whole W discrete symmetry, on the other hand, would lead to four-dimensional theories with smaller gauge symmetry than the one of SM.

4.1.5 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = Sp(4)/(SU(2)\times SU(2))\times(SU(2)/U(1))$. (Case 5e)

Finally, we consider Weyl fermions in the adjoint of $G = E_8$ and the embedding of $R = SU(2)\times SU(2)\times U(1)$ into E_8 suggested by the decomposition

$$\begin{aligned}
E_8 \supset SO(16) &\supset SO(6)\times SO(10) \backsim SU(4)\times SO(10)\\
&\supset (SU^a(2)\times SU^b(2)\times U^I(1))\times SO(10)\\
E_8 \supset &(SU^a(2)\times SU^b(2)\times U^I(1))\times SO(10)\\
\mathbf{248} = &(\mathbf{1},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{1},\mathbf{45})_{(0)} + (\mathbf{3},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{3},\mathbf{1})_{(0)}\\
&+(\mathbf{2},\mathbf{2},\mathbf{1})_{(2)} + (\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)} + (\mathbf{1},\mathbf{1},\mathbf{10})_{(2)} + (\mathbf{1},\mathbf{1},\mathbf{10})_{(-2)}\\
&+(\mathbf{2},\mathbf{2},\mathbf{10})_{(0)} + (\mathbf{2},\mathbf{1},\mathbf{16})_{(1)} + (\mathbf{2},\mathbf{1},\overline{\mathbf{16}})_{(-1)}\\
&+(\mathbf{1},\mathbf{2},\mathbf{16})_{(-1)} + (\mathbf{1},\mathbf{2},\overline{\mathbf{16}})_{(1)}\,.
\end{aligned} \tag{72}$$

If only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU^a(2)\times SU^b(2)\times U^I(1)) = SO(10)\Big(\times U^I(1)\Big)\,.$$

The $R = SU^a(2)\times SU^b(2)\times U^I(1)$ content of vector and spinor of $B_0 = S/R = Sp(4)/(SU^a(2)\times SU^b(2))\times(SU(2)/U(1))$ can be read in the last two columns of table 1. According to the CSDR rules the resulting four-dimensional theory would contain scalars belonging in $\mathbf{10}_{(0)}$, $\mathbf{10}_{(2)}$ and $\mathbf{10}_{(-2)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(1)}$ and $\mathbf{16}_{L(-1)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space $(Sp(4)/SU(2)\times SU(2))\times(SU(2)/U(1))$ (case '**e**' in table 1), are the the center of S, $\mathrm{Z(S)} = (\mathbb{Z}_2)^2$ and the Weyl, $\mathrm{W} = (\mathbb{Z}_2)^2$. According to the list (39) the interesting cases to be examined further are the following.

First, if we employ the Weyl discrete symmetry, $\mathrm{W} = (\mathbb{Z}_2)^2$ leads to a four-dimensional theory with a gauge symmetry described by the group

$$K' = C_H(T^H) = SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\Big(\times U^I(1)\Big)\,.$$

The surviving scalars of the theory belong in

$$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$$

of K', whereas the fermion content of the theory transforms as

$$\begin{aligned}
&(\mathbf{2},\mathbf{1},\mathbf{4})_{L(1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(1)}\,, &&(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1)}\,,\\
&(\mathbf{1},\mathbf{2},\mathbf{4})_{L(1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(1)}\,, &&(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-1)}
\end{aligned} \tag{73}$$

under K'.

Second, if we employ a $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup of the $\mathrm{W}\times\mathrm{Z}(S)$ combination of discrete symmetries, leads to a four-dimensional model with scalars belonging in $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$ of K' and fermions transforming as in eq. (73) but with the signs of the linear combinations reversed.

Finally, in table 8 we also report the less interesting case $\mathbb{Z}_2 \subseteq \mathrm{W}$.

Concerning the spontaneous symmetry breaking of theory, note that for the interesting cases of the $\mathrm{W} = (\mathbb{Z}_2)^2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{W} \times \mathrm{Z}(S)$ discrete symmetries, the final unbroken gauge group in four dimensions is found to be

$$K = SU^{diag}(2) \times SU(4) \Big(\times U(1) \Big).$$

Then, for the case of W discrete symmetry, the fermions of the model transform as

$$\begin{array}{ll} (\mathbf{2},\mathbf{4})_{(1)} - (\mathbf{2},\mathbf{4})'_{(1)}\,, & (\mathbf{2},\mathbf{4})_{(-1)} - (\mathbf{2},\mathbf{4})'_{(-1)}\,, \\ (\mathbf{2},\mathbf{4})_{(1)} + (\mathbf{2},\mathbf{4})'_{(1)}\,, & (\mathbf{2},\mathbf{4})_{(-1)} + (\mathbf{2},\mathbf{4})'_{(-1)}\,, \end{array} \qquad (74)$$

under K, whereas for the case of $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{W} \times \mathrm{Z}(S)$ the fermions belong in similar linear combinations as above but with their signs reversed.

4.2 Dimensional reduction over non-symmetric coset spaces

According to the discussion in sec. 2.4 we have to consider all the possible embeddings $E_8 \supset SO(6) \supset R$, for the six-dimensional *non-symmetric* cosets, S/R, of table 2. It is worth noting that the embedding of R in all cases of six-dimensional non-symmetric cosets are obtained by the following chain of maximal subgroups of $SO(6)$

$$SO(6) \sim SU(4) \supset SU(3) \times U(1) \supset SU(2) \times U(1) \times U(1) \supset U(1) \times U(1) \times U(1)\,. \qquad (75)$$

It is also important to recall from the discussion in secs 2.2 and 2.4 that in all these cases the dimensional reduction of the initial gauge theory leads to an E_6 GUT. The result of our examination in the present section is that the additional use of the Wilson flux breaking mechanism leads to four-dimensional gauge theories based on three different varieties of groups, namely $SO(10) \times U(1)$, $SU(2) \times SU(6)$ or $SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1)$. In the following sections 4.2.1 - 4.2.3 we present details of our examination. We summarize our results in tables 9 and 10 presented in appendix B[‡‡].

4.2.1 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = G_2/SU(3)$. (Case 2a′)

We consider Weyl fermions belonging in the adjoint of $G = E_8$ and identify the R with the $SU(3)$ appearing in the decomposition (62). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU(3)) = E_6\,,$$

i.e. it appears an enhancement of the gauge group, a fact which was noticed earlier in several examples in secs 3.3 and 3.4. This observation suggests that we could have considered the following more obvious embedding of $R = SU(3)$ into E_8,

$$\begin{array}{l} E_8 \supset SU(3) \times E_6 \\ \mathbf{248} = (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{78}) + (\mathbf{3},\mathbf{27}) + (\overline{\mathbf{3}},\overline{\mathbf{27}})\,. \end{array} \qquad (76)$$

‡‡We follow the same notation as in the examination of the symmetric cosets.

The $R = SU(3)$ content of vector and spinor of $B_0 = S/R = G_2/SU(3)$ is $\mathbf{3} + \overline{\mathbf{3}}$ and $\mathbf{1} + \mathbf{3}$, respectively. According to the CSDR rules, the four-dimensional theory would contain scalars belonging in $\mathbf{27}$ and $\overline{\mathbf{27}}$ of $H = E_6$, two copies of chiral fermions transforming as $\mathbf{27}_L$ under the same gauge group and a set of fermions in the $\mathbf{78}$ irrep., since the dimensional reduction over non-symmetric coset preserves the supersymmetric spectrum [14].

The freely acting discrete symmetry, $F^{S/R}$, of the coset space $G_2/SU(3)$ is the Weyl, $\mathrm{W} = \mathbb{Z}_2$ (case '**a**′' in table 2). Then, following the discussion in sec. 3.3.1, the Wilson flux breaking mechanism leads to a four-dimensional theory either with gauge group

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10) \times U(1)\,, \tag{77}$$

in case we embed the $\mathbb{Z}_2$ into the E_6 gauge group as in the embedding (**1**) of sec. 3.3.1, or

$$(ii) \qquad K'^{(\mathbf{2},\mathbf{3})} = C_H(T^H) = SU(2) \times SU(6)\,, \tag{78}$$

in case we choose to embed the discrete symmetry as in the embeddings (**2**) or (**3**) of the same subsection [the superscript in the K''s above refer to the embeddings (**1**), (**2**) or (**3**)].

Making an analysis along the lines presented earlier in the case of symmetric cosets, we determine the particle content of the two models, which is presented in table 10. In both cases the gauge symmetry of the four-dimensional theory cannot be broken further due to the absence of scalars.

4.2.2 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = Sp(4)/(SU(2) \times U(1))_{nonmax}$. (Case 3b′)

We consider Weyl fermions belonging in the adjoint of $G = E_8$ and the decomposition (63). In order the R to be embedded in E_8 as in eq. (35), we identify it with the $SU(2) \times U^I(1)$ appearing in the decomposition (63). Then, if only the CSDR mechanism was applied, the resulting gauge group would be

$$H = C_{E_8}(SU(2) \times U^I(1)) = E_6 \Big(\times U^I(1) \Big)\,. \tag{79}$$

Note that again appears an enhancement of the gauge group. Similarly with previously discussed cases, the additional $U(1)$ factor in the parenthesis corresponds only to a global symmetry. The observation (79) suggests that we could have considered the following embedding of $R = SU(2) \times U(1)$ into E_8[§§],

$$\begin{aligned} &E_8 \supset SU(3) \times E_6 \supset SU(2) \times U^I(1) \times E_6 \\ &\mathbf{248} = (\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{78})_{(0)} + (\mathbf{3},\mathbf{1})_{(0)} + (\mathbf{2},\mathbf{1})_{(-3)} + (\mathbf{2},\mathbf{1})_{(3)} \\ &\quad + (\mathbf{1},\mathbf{27})_{(2)} + (\mathbf{1},\overline{\mathbf{27}})_{(-2)} + (\mathbf{2},\mathbf{27})_{(-1)} + (\mathbf{2},\overline{\mathbf{27}})_{(1)}\,. \end{aligned} \tag{80}$$

The $R = SU(2) \times U^I(1)$ content of vector and spinor of $B_0 = S/R = Sp(4)/(SU(2) \times U^I(1))_{non-max}$ can be read in the last two columns of table 2. According to the CSDR rules then, the surviving scalars in four dimensions would transform as $\mathbf{27}_{(-2)}$, $\mathbf{27}_{(1)}$, $\overline{\mathbf{27}}_{(2)}$ and $\overline{\mathbf{27}}_{(-1)}$ under $H = E_6(\times U^I(1))$. The four-dimensional theory would also contain fermions belonging in $\mathbf{78}_{(0)}$ of H

[§§] This decomposition is in accordance with the Slansky tables but with opposite $U(1)$ charge.

(gaugini of the model), two copies of left-handed fermions belonging in $\mathbf{27}_{L(2)}$ and $\mathbf{27}_{L(-1)}$ and one fermion singlet transforming as $\mathbf{1}_{(0)}$ under the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $Sp(4)/(SU(2)\times U(1))_{non-max}$, are the center of S, Z(S) $=\mathbb{Z}_2$ and the Weyl, W $=\mathbb{Z}_2$. Then, employing the W discrete symmetry, we find that the resulting four-dimensional gauge group is either

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10)\times U(1)\left(\times U^I(1)\right), \qquad \text{or} \tag{81}$$

$$(ii) \qquad K'^{(\mathbf{2},\mathbf{3})} = C_H(T^H) = SU(2)\times SU(6)\left(\times U^I(1)\right), \tag{82}$$

depending on the embedding of $\mathbb{Z}_2 \hookrightarrow E_6$ we choose to consider (see sec. 3.3.1). On the other hand, if we employ the W $\times$ Z(S) $=\mathbb{Z}_2\times\mathbb{Z}_2$ combination of discrete symmetries, the resulting four-dimensional gauge group is either

$$(iii) \qquad K'^{(\mathbf{2}')} = SU(2)\times SU(6)\left(\times U^I(1)\right), \tag{83}$$

in case we embed the $(\mathbb{Z}_2\times\mathbb{Z}_2)$ into the E_6 gauge group as in the embedding $(\mathbf{2}')$ of sec. 3.3.2, or

$$(iv) \qquad K'^{(\mathbf{3}')} = SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)\left(\times U^I(1)\right), \tag{84}$$

in case we choose to embed the discrete symmetry as in the embedding $(\mathbf{3}')$ of the same subsection.

Making a similar analysis as before, we determine the particle content of the four different models, which is presented in table 10. In all cases the gauge symmetry of the resulting four-dimensional theory cannot be broken further by a Higgs mechanism due to the absence of scalars.

4.2.3 Reduction of $G=E_8$ over $B=B_0/F^{B_0}$, $B_0=SU(3)/(U(1)\times U(1))$. (Case 4c$'$)

We consider Weyl fermions in the adjoint of $G=E_8$ and the decomposition (69). In order the $R=U(1)\times U(1)$, to be embedded in E_8 as in eq. (35) one has to identify it with the $U^I(1)\times U^{II}(1)$ appearing in the decomposition (69). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(U^I(1)\times U^{II}(1)) = E_6\left(\times U^I(1)\times U^{II}(1)\right). \tag{85}$$

Note again that an enhancement of the gauge group appears, whereas the additional $U(1)$ factors correspond to global symmetries. The observation (85) suggests that we could have considered the following embedding of $R=U(1)\times U(1)$ into E_8,

$$\begin{aligned}
&E_8 \supset SU(3)\times E_6 \supset (SU(2)\times U^{II}(1))\times E_6 \supset E_6\times U^I(1)\times U^{II}(1)\\
&\quad E_8 \supset E_6\times U^I(1)\times U^{II}(1)\\
&\quad \mathbf{248} = \mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{78}_{(0,0)} + \mathbf{1}_{(-2,0)} + \mathbf{1}_{(2,0)} + \mathbf{1}_{(-1,3)} + \mathbf{1}_{(1,-3)}\\
&\qquad + \mathbf{1}_{(1,3)} + \mathbf{1}_{(-1,-3)} + \mathbf{27}_{(0,-2)} + \overline{\mathbf{27}}_{(0,2)} + \mathbf{27}_{(-1,1)} + \overline{\mathbf{27}}_{(1,-1)}\\
&\qquad + \mathbf{27}_{(1,1)} + \overline{\mathbf{27}}_{(-1,-1)}\,.
\end{aligned} \tag{86}$$

The $R = U^I(1) \times U^{II}(1)$ content of vector and spinor of $B_0 = S/R = SU(3)/(U^I(1) \times U^{II}(1))$ can be read in the last two columns of table 2. The embedding $R \hookrightarrow E_8$ suggested by the decomposition (86) corresponds in the following choice of the $U(1)$ charges appearing in the last case of table 2: $a = 0$, $c = -2$, $b = -1$ and $d = 1$. Then, according to the CSDR rules, the four-dimensional theory would contain scalars which belong in $\mathbf{27}_{(0,-2)}$, $\mathbf{27}_{(0,2)}$, $\mathbf{27}_{(-1,1)}$, $\mathbf{27}_{(1,-1)}$, $\mathbf{27}_{(1,1)}$ and $\mathbf{27}_{(-1,-1)}$ of $H = E_6(\times U^I(1) \times U^{II}(1))$. The resulting four-dimensional theory would also contain gaugini transforming as $\mathbf{78}_{(0,0)}$ under H, two copies of left-handed fermions belonging in $\mathbf{27}_{L(0,-2)}$, $\mathbf{27}_{L(-1,1)}$, $\mathbf{27}_{L(1,1)}$ and two fermion singlets belonging in $\mathbf{1}_{(0,0)}$ and $\mathbf{1}_{(0,0)}$ of the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $SU(3)/(U(1) \times U(1))$ (case '$\mathbf{c}'$' in table 2), are the center of S, Z(S) $= \mathbb{Z}_3$ and the Weyl, W $= \mathbf{S}_3$. Then according to the list (39) only the $\mathbb{Z}_2 \subset$ W discrete symmetry is an interesting case to be examined further.

Then, employing the $\mathbb{Z}_2$ subgroup of the W $= \mathbf{S}_3$ discrete symmetry leads to a four-dimensional theory either with gauge group

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10) \times U(1) \Big(\times U^I(1) \times U^{II}(1) \Big) , \qquad \text{or} \tag{87}$$

$$(ii) \qquad K'^{(\mathbf{2},\mathbf{3})} = C_H(T^H) = SU(2) \times SU(6) \Big(\times U^I(1) \times U^{II}(1) \Big) \tag{88}$$

depending on the embedding of $\mathbb{Z}_2 \hookrightarrow E_6$ we choose to consider (see sec. 3.3.1).

Making a similar analysis as before, we determine the particle content of the two models as follows.

Case (i). The resulting four-dimensional theory contains gaugini which transform as

$$\mathbf{1}_{(0,0,0)} , \qquad \mathbf{45}_{(0,0,0)}$$

under $K'^{(\mathbf{1})}$, a set of fermion singlets which belong in

$$\mathbf{1}_{(0,0,0)} , \qquad \mathbf{1}_{(0,0,0)} ,$$

of $K'^{(\mathbf{1})}$ and a set of chiral fermions which belong in one of the linear combinations

$$\left\{ \begin{array}{l} \mathbf{1}_{L(-4,0,-2)} + \mathbf{1}'_{L(-4,0,-2)} , \\ \mathbf{10}_{L(-2,0,-2)} + \mathbf{10}'_{L(-2,0,-2)} , \\ \mathbf{16}_{L(1,0,-2)} - \mathbf{16}'_{L(1,0,-2)} , \end{array} \right\} , \qquad \left\{ \begin{array}{l} \mathbf{1}_{L(-4,-1,1)} + \mathbf{1}'_{L(-4,-1,1)} , \\ \mathbf{10}_{L(-2,-1,1)} + \mathbf{10}'_{L(-2,-1,1)} , \\ \mathbf{16}_{L(1,-1,1)} - \mathbf{16}'_{L(1,-1,1)} , \end{array} \right\} ,$$

or

$$\left\{ \begin{array}{l} \mathbf{1}_{L(-4,1,1)} + \mathbf{1}'_{L(-4,1,1)} , \\ \mathbf{10}_{L(-2,1,1)} + \mathbf{10}'_{L(-2,1,1)} , \\ \mathbf{16}_{L(1,1,1)} - \mathbf{16}'_{L(1,1,1)} \end{array} \right\}$$

of the same gauge group, depending on the $\mathbb{Z}_2$ subgroup of $\mathbf{S}_3$ that we choose to consider (see table 2).

Case (ii). The resulting four-dimensional theory contains gaugini which transform as

$$(\mathbf{3},\mathbf{1})_{(0,0)}\,, \qquad (\mathbf{1},\mathbf{35})_{(0,0)}$$

under $K'^{(\mathbf{2},\mathbf{3})}$, a set of fermion singlets which belong in

$$(\mathbf{1},\mathbf{1})_{(0,0)}\,, \qquad (\mathbf{1},\mathbf{1})_{(0,0)}\,,$$

of $K'^{(\mathbf{2},\mathbf{3})}$ and a set of chiral fermions which belong in one of the linear combinations

$$\left\{\begin{array}{ll} (\mathbf{1},\mathbf{15})_{L(0,-2)} + (\mathbf{1},\mathbf{15})'_{L(0,-2)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(0,-2)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(0,-2)}\,, \end{array}\right\}, \qquad \left\{\begin{array}{ll} (\mathbf{1},\mathbf{15})_{L(-1,1)} + (\mathbf{1},\mathbf{15})'_{L(-1,1)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(-1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(-1,1)}\,, \end{array}\right\},$$

or

$$\left\{\begin{array}{ll} (\mathbf{1},\mathbf{15})_{L(1,1)} + (\mathbf{1},\mathbf{15})'_{L(1,1)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(1,1)}\,, \end{array}\right\}$$

of the same gauge group, depending on the $\mathbb{Z}_2$ subgroup of $\mathbf{S}_3$ that we choose to consider (see table 2).

Note that in both cases the gauge symmetry of the four-dimensional theory cannot be broken further by a Higgs mechanism due to the absence of scalars.

Finally, if we have used either the symmetric group of 3 permutations, $\mathbf{S}_3$, or its subgroup $\mathbb{Z}_3 \subset \mathbf{S}_3$, we could not form linear combinations among the two copies of the CSDR-surviving left-handed fermions and no fermions would survive in four dimensions.

5 Conclusions

The CSDR is a consistent dimensional reduction scheme [57], as well as an elegant framework to incorporate in a unified manner the gauge and the ad-hoc Higgs sector of spontaneously broken four-dimensional gauge theories using the extra dimensions. The kinetic terms of fermions were easily included in the same unified description. A striking feature of the scheme concerning fermions was the discovery that chiral ones can be introduced [10] and moreover they could result even from vector-like reps of the higher dimensional gauge theory [3,11]. This possibility is due to the presence of non-trivial background gauge configurations required by the CSDR principle, in accordance with the index theorem. Another striking feature of the theory is the possibility that the softly broken sector of the four-dimensional supersymmetric theories can result from a higher-dimensional $\mathcal{N}=1$ supersymmetric gauge theory with only a vector supermultiplet, when is dimensionally reduced over non-symmetric coset spaces [14]. Another interesting feature useful in realistic model searches is the possibility to deform the metric in certain non-symmetric coset spaces and introduce more than one scales [3, 8].

Recently there exist a revival of interest in the study of compactifications with internal manifolds six-dimensional non-symmetric coset spaces possessing an $SU(3)$-structure within the framework of flux compactifications. Motivated by this interest we plan to examine the CSDR of the heterotic ten-dimensional gauge theory in successive steps. In the present work, starting with a supersymmetric

$\mathcal{N}=1$, E_8 gauge theory in ten dimensions we made a complete classification of the models obtained in four dimensions after reducing the theory over all multiply connected six-dimensional coset spaces, resulting by moding out all the freely acting discrete symmetries on these manifolds, and using the Wilson flux breaking mechanism in an exhaustive way. The results of our extended investigation have been partially presented in a short communication [58]. Despite some partial success, our result is that the two mechanisms used to break the gauge symmetry, i.e. the geometric breaking of the CSDR and the topological of the Hosotani mechanism are not enough to lead the four-dimensional theory to the SM or some interesting extension as the MSSM. Limiting ourselves in the old CSDR framework one can think of some new sources of gauge symmetry breaking, such as new scalars coming from a gauge theory defined even in higher dimensions [59, 60]. Much more interesting is to extend our examination in a future study of the full ten-dimensional $E_8 \times E_8$ gauge theory of the heterotic string. Moreover in that case one does not have to be restricted in the study of freely acting discrete symmetries of the coset spaces and can extent the analysis including orbifolds [61–63]. More possibilities are offered in refs [64].

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Appendices

A Dimensional reduction over symmetric $6D$ coset spaces

Table 7: **Dimensional reduction over symmetric $6D$ coset spaces**. *Particle physics models leading to $SO(10)$ GUTs in four dimensions.*

Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
1a	$E_8 \supset SO(6)\times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{45})+(\mathbf{6},\mathbf{10})+(\mathbf{15},\mathbf{1})$ $+(\mathbf{4},\mathbf{16})+(\overline{\mathbf{4}},\overline{\mathbf{16}})$	$\frac{SO(7)}{SO(6)}$	$SO(10)$	$\mathbf{10}$	$\mathbf{16}_L$ $\mathbf{16}'_L$
2b	$E_8 \supset SO(6)\times SO(10)$ $SO(6) \backsim SU(4) \supset SU(3)\times U^I(1)$ $E_8 \supset (SU(3)\times U^I(1))\times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{45})_{(0)}+(\mathbf{8},\mathbf{1})_{(0)}$ $+(\mathbf{3},\mathbf{10})_{(-2)}+(\overline{\mathbf{3}},\mathbf{10})_{(2)}$ $+(\mathbf{3},\mathbf{1})_{(4)}+(\overline{\mathbf{3}},\mathbf{1})_{(-4)}$ $+(\mathbf{1},\mathbf{16})_{(-3)}+(\mathbf{1},\overline{\mathbf{16}})_{(3)}$ $+(\mathbf{3},\mathbf{16})_{(1)}+(\overline{\mathbf{3}},\overline{\mathbf{16}})_{(-1)}$	$\frac{SU(4)}{SU(3)\times U^I(1)}$	$SO(10)\left(\times U^I(1)\right)$	$\mathbf{10}_{(-2)}$ $\mathbf{10}_{(2)}$	$\mathbf{16}_{L(3)}$ $\mathbf{16}_{L(-1)}$ $\mathbf{16}'_{L(3)}$ $\mathbf{16}'_{L(-1)}$

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Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
3d, **6d**	$E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4) \supset SU'(3) \times U^{II}(1)$ $SU'(3) \supset SU^a(2) \times U^I(1)$ $Y^{I'} = \frac{b}{3} Y^I + \frac{b}{3} Y^{II}$ $Y^{II'} = \frac{2a}{3} Y^I - \frac{a}{3} Y^{II}$ **or** $E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4)$ $SU(4) \supset SU^a(2) \times SU^b(2) \times U^{II}(1)$ $SU^b(2) \supset U^I(1)$ $Y^{I'} = -b\, Y^I$ $Y^{II'} = a\, Y^{II}$ $E_8 \supset (SU^a(2) \times U^{I'}(1) \times U^{II'}(1)) \times SO(10)$ $\mathbf{248} = (\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1})_{(0,0)}$ $+ (\mathbf{3},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{45})_{(0,0)}$ $+ (\mathbf{1},\mathbf{1})_{(-2b,0)} + (\mathbf{1},\mathbf{1})_{(2b,0)}$ $+ (\mathbf{2},\mathbf{1})_{(-b,2a)} + (\mathbf{2},\mathbf{1})_{(b,-2a)}$ $+ (\mathbf{2},\mathbf{1})_{(-b,-2a)} + (\mathbf{2},\mathbf{1})_{(b,2a)}$ $+ (\mathbf{1},\mathbf{10})_{(0,-2a)} + (\mathbf{1},\mathbf{10})_{(0,2a)}$ $+ (\mathbf{2},\mathbf{10})_{(b,0)} + (\mathbf{2},\mathbf{10})_{(-b,0)}$ $+ (\mathbf{1},\mathbf{16})_{(b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(-b,a)}$ $+ (\mathbf{1},\mathbf{16})_{(-b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(b,a)}$ $+ (\mathbf{2},\mathbf{16})_{(0,a)} + (\mathbf{2},\overline{\mathbf{16}})_{(0,-a)}$	$\left(\frac{SU(3)}{SU^a(2) \times U^{I'}(1)}\right) \times \left(\frac{SU(2)}{U^{II'}(1)}\right)$	$SO(10)$ $\left(\times U^{I'}(1) \times U^{II'}(1)\right)$	$\mathbf{10}_{(0,-2a)}$ $\mathbf{10}_{(0,2a)}$ $\mathbf{10}_{(b,0)}$ $\mathbf{10}_{(-b,0)}$	$\mathbf{16}_{L(b,-a)}$ $\mathbf{16}_{L(-b,-a)}$ $\mathbf{16}_{L(0,a)}$ $\mathbf{16}'_{L(b,-a)}$ $\mathbf{16}'_{L(-b,-a)}$ $\mathbf{16}'_{L(0,a)}$

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Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
4f, 7f	$E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4) \supset SU'(3) \times U^{III}(1)$ $SU'(3) \supset SU^a(2) \times U^{II}(1)$ $SU^a(2) \supset U^I(1)$ $Y^{I'} = aY^I + \frac{a}{3}Y^{II} + \frac{a}{3}Y^{III}$ $Y^{II'} = -bY^I + \frac{b}{3}Y^{II} + \frac{b}{3}Y^{III}$ $Y^{III'} = -\frac{2c}{3}Y^I + \frac{c}{3}Y^{III}$ **or** $E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4)$ $SU(4) \supset SU^a(2) \times SU^b(2) \times U^{III}(1)$ $SU^a(2) \supset U^{II}(1)$ $SU^b(2) \supset U^I(1)$ $Y^{I'} = a\,Y^I - a\,Y^{II}$ $Y^{II'} = -b\,Y^I - b\,Y^{II}$ $Y^{III'} = -c\,Y^{III}$ $E_8 \supset SO(10)$ $\times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1)$ $\mathbf{248} = \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)}$ $+\mathbf{45}_{(0,0,0)}$ $+\mathbf{1}_{(-2a,2b,0)} + \mathbf{1}_{(2a,-2b,0)}$ $+\mathbf{1}_{(-2a,-2b,0)} + \mathbf{1}_{(2a,2b,0)}$ $+\mathbf{1}_{(-2a,0,-2c)} + \mathbf{1}_{(2a,0,2c)}$ $+\mathbf{1}_{(0,-2b,-2c)} + \mathbf{1}_{(0,2b,2c)}$ $+\mathbf{1}_{(-2a,0,2c)} + \mathbf{1}_{(2a,0,-2c)}$ $+\mathbf{1}_{(0,-2b,2c)} + \mathbf{1}_{(0,2b,-2c)}$ $+\mathbf{10}_{(0,0,2c)} + \mathbf{10}_{(0,0,-2c)}$ $+\mathbf{10}_{(0,2b,0)} + \mathbf{10}_{(0,-2b,0)}$ $+\mathbf{10}_{(2a,0,0)} + \mathbf{10}_{(-2a,0,0)}$ $+\mathbf{16}_{(a,b,c)} + \overline{\mathbf{16}}_{(-a,-b,-c)}$ $+\mathbf{16}_{(-a,-b,c)} + \overline{\mathbf{16}}_{(a,b,-c)}$ $+\mathbf{16}_{(-a,b,-c)} + \overline{\mathbf{16}}_{(a,-b,c)}$ $+\mathbf{16}_{(a,-b,-c)} + \overline{\mathbf{16}}_{(-a,b,c)}$	$\left(\frac{SU(2)}{U^{I'}(1)}\right) \times \left(\frac{SU(2)}{U^{II'}(1)}\right)$ $\times \left(\frac{SU(2)}{U^{III'}(1)}\right)$	$SO(10) \left(\times U^{I'}(1) \right.$ $\left. \times U^{II'}(1) \times U^{III'}(1) \right)$	$\mathbf{10}_{(2a,0,0)}$ $\mathbf{10}_{(-2a,0,0)}$ $\mathbf{10}_{(0,2b,0)}$ $\mathbf{10}_{(0,-2b,0)}$ $\mathbf{10}_{(0,0,2c)}$ $\mathbf{10}_{(0,0,-2c)}$	$\mathbf{16}_{L(a,b,c)}$ $\mathbf{16}_{L(-a,-b,c)}$ $\mathbf{16}_{L(-a,b,-c)}$ $\mathbf{16}_{L(a,-b,-c)}$ $\mathbf{16}'_{L(a,b,c)}$ $\mathbf{16}'_{L(-a,-b,c)}$ $\mathbf{16}'_{L(-a,b,-c)}$ $\mathbf{16}'_{L(a,-b,-c)}$
5e	$E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4)$ $SU(4) \supset SU^a(2) \times SU^b(2) \times U^I(1)$ $E_8 \supset (SU^a(2) \times SU^b(2) \times U^I(1))$ $\times SO(10)$ $\mathbf{248} = (\mathbf{1},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{1},\mathbf{45})_{(0)}$ $+(\mathbf{3},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{3},\mathbf{1})_{(0)}$ $+(\mathbf{2},\mathbf{2},\mathbf{1})_{(2)} + (\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)}$ $+(\mathbf{1},\mathbf{1},\mathbf{10})_{(2)} + (\mathbf{1},\mathbf{1},\mathbf{10})_{(-2)}$ $+(\mathbf{2},\mathbf{2},\mathbf{10})_{(0)}$ $+(\mathbf{2},\mathbf{1},\mathbf{16})_{(1)} + (\mathbf{2},\mathbf{1},\overline{\mathbf{16}})_{(-1)}$ $+(\mathbf{1},\mathbf{2},\mathbf{16})_{(-1)} + (\mathbf{1},\mathbf{2},\overline{\mathbf{16}})_{(1)}$	$\frac{Sp(4) \times SU(2)}{SU^a(2) \times SU^b(2) \times U^I(1)}$	$SO(10) \left(\times U^I(1) \right)$	$\mathbf{10}_{(0)}$ $\mathbf{10}_{(2)}$ $\mathbf{10}_{(-2)}$	$\mathbf{16}_{L(1)}$ $\mathbf{16}_{L(-1)}$ $\mathbf{16}'_{L(1)}$ $\mathbf{16}'_{L(-1)}$

Table 8: **Application of Hosotani breaking mechanism on particle physics models which are listed in table 7**.

Case	Discrete Symmetries	$\mathbf{K}'$	Surviving scalars under $\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$	$\mathbf{K}$	Surviving fermions under $\mathbf{K}$
1a	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$	$(\mathbf{1},\mathbf{1},\mathbf{6})$	$(\mathbf{2},\mathbf{1},\mathbf{4})_L-(\mathbf{2},\mathbf{1},\mathbf{4})'_L$ $(\mathbf{1},\mathbf{2},\mathbf{4})_L+(\mathbf{1},\mathbf{2},\mathbf{4})'_L$	Not Interesting	
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$)	„	$(\mathbf{2},\mathbf{2},\mathbf{1})$	$+\leftrightarrow-$	$SU^{diag}(2)\times SU(4)$	$(\mathbf{2},\mathbf{4})_L+(\mathbf{2},\mathbf{4})'_L$ $(\mathbf{2},\mathbf{4})_L-(\mathbf{2},\mathbf{4})'_L$
2b	W ($\mathbb{1}$)	H Unbroken				
	W × Z ($\mathbb{Z}_4$)	Not Interesting				
3d, **6d**	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(-b,0)}$	$(\mathbf{2},\mathbf{1},\mathbf{4})_{L(b,-a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(b,-a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-b,-a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-b,-a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(0,a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(0,a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(0,a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(0,a)}$	Not Interesting	
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$)	„	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(-b,0)}$	$+\leftrightarrow-$	$SU^{diag}(2)\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)\Big)$	$(\mathbf{2},\mathbf{4})_{L(b,-a)}+(\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(b,-a)}-(\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(-b,-a)}+(\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(-b,-a)}-(\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(0,a)}+(\mathbf{2},\mathbf{4})'_{L(0,a)}$ $(\mathbf{2},\mathbf{4})_{L(0,a)}-(\mathbf{2},\mathbf{4})'_{L(0,a)}$
4f, **7f**	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)$ $\times U^{III'}(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,2b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,-2b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)}$	$-$	Not Interesting	
	W ($(\mathbb{Z}_2)^2$)	„	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)}$	„	„	
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$)	„	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,2b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,-2b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,2c)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,-2c)}$	„	„	
5e	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0)}$	$(\mathbf{2},\mathbf{1},\mathbf{4})_{L(1)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(1)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(1)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-1)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-1)}$	Not Interesting	

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Case	**Discrete Symmetries**	$\mathbf{K}'$	**Surviving scalars under** $\mathbf{K}'$	**Surviving fermions under** $\mathbf{K}'$	$\mathbf{K}$	**Surviving fermions under** $\mathbf{K}$
	W $(\mathbb{Z}_2)^2$	”	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$	”	$SU^{diag}(2)\times SU(4)$ $\Big(\times U(1)\Big)$	$(\mathbf{2},\mathbf{4})_{L(1)}-(\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(1)}+(\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)}-(\mathbf{2},\mathbf{4})'_{L(-1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)}+(\mathbf{2},\mathbf{4})'_{L(-1)}$
	W $\times$ Z $(\mathbb{Z}_2\times\mathbb{Z}_2)$	”	”	$+\leftrightarrow-$	”	$(\mathbf{2},\mathbf{4})_{L(1)}+(\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(1)}-(\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)}+(\mathbf{2},\mathbf{4})'_{L(-1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)}-(\mathbf{2},\mathbf{4})'_{L(-1)}$

B Dimensional reduction over non-symmetric $6D$ coset spaces

Table 9: **Dimensional reduction over symmetric $6D$ coset spaces**. *Particle physics models leading to E_6 GUTs in four dimensions.*

Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
2a′	$E_8 \supset SU'(3) \times E_6$ $E_8 \supset SU'(3) \times E_6$ $\mathbf{248}=(\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{78})$ $+(\mathbf{3},\mathbf{27})+(\overline{\mathbf{3}},\overline{\mathbf{27}})$	$\frac{G_2}{SU'(3)}$	E_6	$\mathbf{27}$ $\overline{\mathbf{27}}$	$\mathbf{78}$ $\mathbf{27}_L$ $\mathbf{27}'_L$
3b′	$E_8 \supset SU'(3) \times E_6$ $SU'(3) \supset SU^a(2) \times U^I(1)$ $Y^{I'} = -Y^I$ $E_8 \supset SU^a(2) \times U^{I'}(1) \times E_6$ $\mathbf{248}=(\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{78})_{(0)}$ $+(\mathbf{3},\mathbf{1})_{(0)}$ $+(\mathbf{2},\mathbf{1})_{(-3)}+(\mathbf{2},\mathbf{1})_{(3)}$ $+(\mathbf{1},\mathbf{27})_{(2)}+(\mathbf{1},\overline{\mathbf{27}})_{(-2)}$ $+(\mathbf{2},\mathbf{27})_{(-1)}+(\mathbf{2},\overline{\mathbf{27}})_{(1)}$	$\left(\frac{Sp(4)}{SU^a(2)\times U^I(1)}\right)_{nonmax}$	$E_6\left(\times U^{I'}(1)\right)$	$\mathbf{27}_{(2)}$ $\mathbf{27}_{(-1)}$ $\overline{\mathbf{27}}_{(-2)}$ $\overline{\mathbf{27}}_{(1)}$	$\mathbf{1}_{(0)}$ $\mathbf{78}_{(0)}$ $\mathbf{27}_{L(2)}$ $\mathbf{27}_{L(-1)}$ $\mathbf{27}'_{L(2)}$ $\mathbf{27}'_{L(-1)}$
4c′	$E_8 \supset SU'(3) \times E_6$ $SU'(3) \supset SU^a(2) \times U^{II}(1)$ $SU^a(2) \supset U^I(1)$ $E_8 \supset E_6 \times U^I(1) \times U^{II}(1)$ $\mathbf{248}=\mathbf{1}_{(0,0)}+\mathbf{1}_{(0,0)}$ $+\mathbf{78}_{(0)}$ $+\mathbf{1}_{(-2,0)}+\mathbf{1}_{(2,0)}$ $+\mathbf{1}_{(-1,3)}+\mathbf{1}_{(1,-3)}$ $+\mathbf{1}_{(1,3)}+\mathbf{1}_{(-1,-3)}$ $+\mathbf{27}_{(0,-2)}+\overline{\mathbf{27}}_{(0,2)}$ $+\mathbf{27}_{(-1,1)}+\overline{\mathbf{27}}_{(1,-1)}$ $+\mathbf{27}_{(1,1)}+\overline{\mathbf{27}}_{(-1,-1)}$	$\frac{SU(3)}{U^I(1)\times U^{II}(1)}$	$E_6\left(\times U^I(1)\times U^{II}(1)\right)$	$\mathbf{27}_{(0,-2)}$ $\mathbf{27}_{(-1,1)}$ $\mathbf{27}_{(1,1)}$ $\overline{\mathbf{27}}_{(0,2)}$ $\overline{\mathbf{27}}_{(1,-1)}$ $\overline{\mathbf{27}}_{(-1,-1)}$ $(a{=}0,c{=}-2)$ $(b{=}-1,d{=}1)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $\mathbf{78}_{(0,0)}$ $\mathbf{27}_{L(0,-2)}$ $\mathbf{27}_{L(-1,1)}$ $\mathbf{27}_{L(1,1)}$ $\mathbf{27}'_{L(0,-2)}$ $\mathbf{27}'_{L(-1,1)}$ $\mathbf{27}'_{L(1,1)}$

Table 10: **Application of Hosotani breaking mechanism on particle physics models which are listed in table 9.** *The surviving fields are calculated for the embeddings $\mathbb{Z}_2 \hookrightarrow E_6$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \hookrightarrow E_6$, discussed in subsections 3.3.1 and 3.3.2*

Case	Discrete Symmetries	K′	Surviving fermions under K′
2a′	W ($\mathbb{Z}_2$) [*embedding* (**1**)]	$SO(10) \times U(1)$	$\mathbf{1}_{(0)}$ $\mathbf{45}_{(0)}$ $\mathbf{1}_{L(-4)}+\mathbf{1}'_{L(-4)}$ $\mathbf{10}_{L(-2)}+\mathbf{10}'_{L(-2)}$ $\mathbf{16}_{L(1)}-\mathbf{16}'_{L(1)}$
	W ($\mathbb{Z}_2$) [*embeddings* (**2**), (**3**)]	$SU(2) \times SU(6)$	$(\mathbf{3},\mathbf{1})$ $(\mathbf{1},\mathbf{35})$ $(\mathbf{1},\mathbf{15})_L+(\mathbf{1},\mathbf{15})'_L$ $(\mathbf{2},\overline{\mathbf{6}})_L-(\mathbf{2},\overline{\mathbf{6}})'_L$

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Case	Discrete Symmetries	$\mathbf{K'}$	Surviving fermions under $\mathbf{K'}$
$\mathbf{3b'}$	W $(\mathbb{Z}_2)$ [*embedding* $(\mathbf{1})$]	$SO(10)\times U(1)\left(\times U^I(1)\right)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $\mathbf{45}_{(0,0)}$ $\mathbf{1}_{L(-4,2)}+\mathbf{1}'_{L(-4,2)}$ $\mathbf{10}_{L(-2,2)}+\mathbf{10}'_{L(-2,2)}$ $\mathbf{16}_{L(1,2)}-\mathbf{16}'_{L(1,2)}$ $\mathbf{1}_{L(-4,-1)}+\mathbf{1}'_{L(-4,-1)}$ $\mathbf{10}_{L(-2,-1)}+\mathbf{10}'_{L(-2,-1)}$ $\mathbf{16}_{L(1,-1)}-\mathbf{16}'_{L(1,-1)}$
	W $(\mathbb{Z}_2)$ [*embeddings* $(\mathbf{2})$, $(\mathbf{3})$]	$SU(2)\times SU(6)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1})_{(0)}$ $(\mathbf{3},\mathbf{1})_{(0)}$ $(\mathbf{1},\mathbf{35})_{(0)}$ $(\mathbf{1},\mathbf{15})_{L(2)}+(\mathbf{1},\mathbf{15})'_{L(2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(2)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(2)}$ $(\mathbf{1},\mathbf{15})_{L(-1)}+(\mathbf{1},\mathbf{15})'_{L(-1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(-1)}$
	W $\times$ Z $(\mathbb{Z}_2\times\mathbb{Z}_2)$ [*embedding* $(\mathbf{2'})$]	$SU(2)\times SU(6)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1})_{(0)}$ $(\mathbf{3},\mathbf{1})_{(0)}$ $(\mathbf{1},\mathbf{35})_{(0)}$ $(\mathbf{1},\mathbf{15})_{L(2)}-(\mathbf{1},\mathbf{15})'_{L(2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(2)}+(\mathbf{2},\overline{\mathbf{6}})'_{L(2)}$ $(\mathbf{1},\mathbf{15})_{L(-1)}-(\mathbf{1},\mathbf{15})'_{L(-1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1)}+(\mathbf{2},\overline{\mathbf{6}})'_{L(-1)}$
	W $\times$ Z $(\mathbb{Z}_2\times\mathbb{Z}_2)$ [*embedding* $(\mathbf{3'})$]	$SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{3},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{3},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{15})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{L(4,2)}+(\mathbf{1},\mathbf{1},\mathbf{1})'_{L(4,2)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{L(2,2)}+(\mathbf{2},\mathbf{2},\mathbf{1})'_{L(2,2)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{L(-2,2)}+(\mathbf{1},\mathbf{1},\mathbf{6})'_{L(-2,2)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1,2)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1,2)}$ $(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,2)}-(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,2)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{L(4,-1)}+(\mathbf{1},\mathbf{1},\mathbf{1})'_{L(4,-1)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{L(2,-1)}+(\mathbf{2},\mathbf{2},\mathbf{1})'_{L(2,-1)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{L(-2,-1)}+(\mathbf{1},\mathbf{1},\mathbf{6})'_{L(-2,-1)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1,-1)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1,-1)}$ $(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,-1)}-(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,-1)}$

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Case	Discrete Symmetries	$\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$
$\mathbf{4c}'$	W $(\mathbb{Z}_2)$ [*embedding* $(\mathbf{1})$]	$SO(10)\times U(1)\left(\times U^{I}(1)\times U^{II}(1)\right)$	$\mathbf{1}_{(0,0,0)}$ $\mathbf{1}_{(0,0,0)}$ $\mathbf{1}_{(0,0,0)}$ $\mathbf{45}_{(0,0,0)}$ and $\mathbf{1}_{L(-4,0,-2)}+\mathbf{1}'_{L(-4,0,-2)}$ $\mathbf{10}_{L(-2,0,-2)}+\mathbf{10}'_{L(-2,0,-2)}$ $\mathbf{16}_{L(1,0,-2)}-\mathbf{16}'_{L(1,0,-2)}$ or $\mathbf{1}_{L(-4,-1,1)}+\mathbf{1}'_{L(-4,-1,1)}$ $\mathbf{10}_{L(-2,-1,1)}+\mathbf{10}'_{L(-2,-1,1)}$ $\mathbf{16}_{L(1,-1,1)}-\mathbf{16}'_{L(1,-1,1)}$ or $\mathbf{1}_{L(-4,1,1)}+\mathbf{1}'_{L(-4,1,1)}$ $\mathbf{10}_{L(-2,1,1)}+\mathbf{10}'_{L(-2,1,1)}$ 1 $\mathbf{16}_{L(1,1,1)}-\mathbf{16}'_{L(1,1,1)}$
	W $(\mathbb{Z}_2)$ [*embeddings* $(\mathbf{2})$, $(\mathbf{3})$]	$SU(2)\times SU(6)\left(\times U^{I}(1)\times U^{II}(1)\right)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $(\mathbf{3},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{35})_{(0,0)}$ and $(\mathbf{1},\mathbf{15})_{L(0,-2)}+(\mathbf{1},\mathbf{15})'_{L(0,-2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(0,-2)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(0,-2)}$ or $(\mathbf{1},\mathbf{15})_{L(-1,1)}+(\mathbf{1},\mathbf{15})'_{L(-1,1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1,1)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(-1,1)}$ or $(\mathbf{1},\mathbf{15})_{L(1,1)}+(\mathbf{1},\mathbf{15})'_{L(1,1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(1,1)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(1,1)}$

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